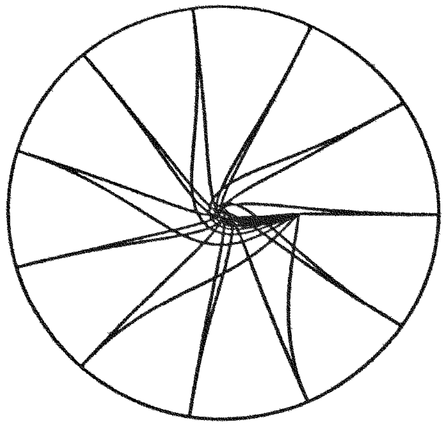
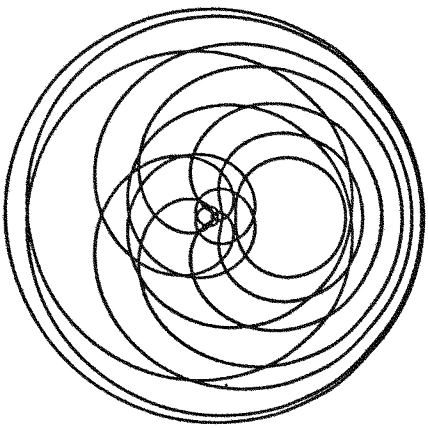
COEFFICIENT ESTIMATES FOR BOUNDED NONVANISHING FUNCTIONS





RUUD ERMERS

COEFFICIENT ESTIMATES FOR
BOUNDED NONVANISHING FUNCTIONS

COEFFICIENT ESTIMATES FOR BOUNDED NONVANISHING FUNCTIONS

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Aan mijn ouders

Contents

Preface and Summary

In this thesis we consider several problems regarding the Taylor coefficients of functions, analytic in the unit disk. One of the main topics in complex function theory (of one variabele) is the estimation of coefficients of functions in a given class. A famous example is Bieberbach's conjecture, dating from 1916, which asserts that the Taylor coefficients of each function of class **S** satisfy the inequality $|a_n| \leq n$. We consider several classes of *bounded* analytic functions omitting a certain value.

If we know that in a given class W, all functions are uniformally bounded, i.e. there exists $M > 0$ such that $|f(z)| \leq M$ $(z \in \Delta_1,\ f \in W)$, then it follows that the coefficients of each $f \in W$ satisfy $|a_n| \leq M$.

In case we have no other constraints on the functions in the class W, it immediately follows that the bound $|a_n| \leq M$ is sharp, i.e. there exists $f \in W$ (namely $f : z \to Mz^n$) for which we have equality. It is natural to ask what happens if we have other constraints on W.

The class $\mathbf{B}^*$ is defined as

$$\mathbf{B}^* = \{f \in H(\Delta_1) \mid 0 < |f| \leq 1\}$$

so the additional constraint is here that $f \in \mathbf{B}^*$ does not vanish.

The problem of finding sharp upper bounds for the coefficients is not easy. Krzyż conjectured that $|a_n| \leq \frac{2}{e}$ $(n \in \mathbf{N})$. We prove this conjecture for the first four coefficients. Furthermore, we consider several other classes of functions, all closely related to the class $\mathbf{B}^*$.

Several techniques are used, such as the subordination principle, Schur's method (in a revised version) and variational methods. Furthermore, some estimates are derived by using some heavy, but elementary calculus:

Since the problem of finding the sharp estimate for the Krzyż-conjecture may be viewed as a *nonlineair* coefficient problem over the class **P** it follows that the calculations involved become rapidly fairly complex. This seems to be the nature of the problem. When using a computer one easily checks that some inequalities must hold. Unfortunately, proving them can sometimes involve some heavy calculations.

We believe that using a computer *can* help solving complex problems, but we have also experienced that a careful analysis of the problem reduces its use to the right proportions.

For example, see Theorems 2.5 and 2.6. At first we were only able to prove them using some mysterious calculus, obtained with a computer. Fortunately, after many discussions, we were able to dispose part of the proof, replacing it by an application of (the beautiful) Lemma 2.5.

Now, almost all computer calculations have been removed, in favour of more elegant methods, except perhaps for the Krzyż-conjecture in case $n = 4$. We hope to return to that topic in the near future.

In chapter 1 several tools are introduced; they will be needed in the rest of the thesis. Most of them, such as the subordination principle and variational methods, are well-known. We introduce them in order to keep this thesis as much self-contained as possible. Some classes of functions, related to the class $\mathbf{B}^*$ are introduced, including the well-known classes $\mathbf{P}$ and $\mathbf{S}$. We prove several theorems concerning relations between the various classes. Here again, we have tried to keep the proofs self-contained.

Furthermore, we show how we can restate Schur's method (a result of Campschroer [4]); This method will be the key to the proofs given for the Krzyż-conjecture.

In chapter 2 we consider the class $\mathbf{B}^*$ in depth. We characterize the extremal functions for the Krzyż-conjecture. This characterization is then used to deduce a uniform upper bound for the coefficients of functions in $\mathbf{B}^*$. We develop a method to obtain a sharp estimate for the first two coefficients of functions in the class $\mathbf{B}^*_\gamma$ which is also used to prove a theorem concerning early coefficients of functions f which map Δ_1 into an annulus. The rest of the chapter is devoted to the Krzyż-conjecture for the first four coefficients.

In chapter 3 we consider the subclass of $\mathbf{B}^*$, consisting of the univalent elements. The extra constraint of univalency makes things much more difficult. We can only give sharp upper bounds for the first two coefficients. The proof concerning the second coefficient is quite elaborate, but elementary. It is based on the proof given by Prokhorov and Szynal [24]. Their proof is computer-based, but by considering things more carefully, we could eliminate *all* computer calculations. In the second part of the chapter we give an estimate for the third coefficient of univalent $\mathbf{B}^*$ elements, having real coefficients. The method used here, is based on the work of Tammi [29].

Finally, in chapter 4 we consider the subclass of $\mathbf{B}^*$, consisting of the polynomial elements. We prove that this class is a dense subclass of $\mathbf{B}^*$. Working with polynomials has the advantage that one can obtain information about the coefficients from the location of the zeros of the functions. We consider the subclass $\mathbf{P}^\Gamma$ containing only those polynomials who have all their zeros on Γ_1. We prove (part of) a conjecture raised by Saff and Sheil-Small which relates the coefficients of a $\mathbf{P}^\Gamma$ function to the coefficients of a $\mathbf{B}^*$ function. The last part of this chapter is devoted to polynomial elements of degree 3, with real coefficients. We give sharp upper bounds for the coefficients of those functions. It turns out that the extremal functions are quite unexpected.

List of Symbols

Notation	Short description	page
Δ_1	$\{z \in \mathbf{C} \mid \lvert z\rvert < 1\}$	-
Γ_1	$\{z \in \mathbf{C} \mid \lvert z\rvert = 1\}$	-
$H(\Delta_1)$	$\{f \mid f$ is holomorphic in $\Delta_1\}$	1
$\mathcal{A}_n(W)$	$\sup\limits_{f\in W} \left\lvert \frac{f^{(n)}(0)}{n!} \right\rvert$	1
$\mathbf{B}^*$	$\{f \in H(\Delta_1) \mid 0 < \lvert f\rvert \leq 1\}$	1
$\mathbf{B}_0$	$\{w \in H(\Delta_1) \mid w(0) = 0, \lvert w\rvert < 1\}$	4
$M_p(r, f)$	$\left(\frac{1}{2\pi}\int_0^{2\pi} \lvert f(re^{i\theta})\rvert^p \, d\theta\right)^{\frac{1}{p}}$	4
$\mathbf{P}$	$\{f \in H(\Delta_1) \mid f(0) = 1,\ \Re e\, f > 0\}$	5
$\mathbf{B}$	$\{f \in H(\Delta_1) \mid \lvert f\rvert \leq 1\}$	8
$\mathbf{S}$	$\{f \in H(\Delta_1) \mid f(0) = 0,\ f'(0) = 1,\ f$ is univalent $\}$	10
$\langle x, y\rangle$	$\sum\limits_{i=1}^{n} x_i\overline{y_i}$	11
$\ll$	Majorization in Schur's sense	12
$C(\mathbf{B}^*)$	$\sup\limits_{n\in\mathbf{N}} \mathcal{A}_n(\mathbf{B}^*)$	19
$\mu(M)$	The Lebesgue measure of M	22
$\mathbf{B}^*_\gamma$	$\{f \in \mathbf{B} \mid f$ omits some $w, \lvert w\rvert = \gamma\}$	27
$\mathbf{B}^*_U$	$\{f \in \mathbf{B}^* \mid f$ is univalent$\}$	49

Notation	Short description	page
k_d	The univalent function which maps Δ_1 onto $\{z \mid \lvert z\rvert < 1, z \notin (-1,0]\}$, with $k_d(0) = d$, $k'_d(0) > 0$	49
$\mathbf{B}^*_U(\mathbf{R})$	$\{f \in \mathbf{B}^*_U \mid f$ has real coefficients$\}$	57
$\mathbf{S}_{\mathbf{R}}(b)$	$\{f \in \mathbf{B} \mid f(0) = 0, f'(0) = b,$ f is univalent and has real coefficients$\}$	60
$\mathbf{P}^*$	$\{p \in \mathbf{B}^* \mid p$ is a polynomial$\}$	67
$\mathbf{P}^\Gamma$	$\{p \in \mathbf{P}^* \mid p$ has all its zeros on $\Gamma_1\}$	67
$\mathbf{P}^*_m$	$\{p \in \mathbf{B}^* \mid p$ is a polynomial of degree exactly $m\}$	67
$\mathbf{P}^\Gamma_m$	$\{p \in \mathbf{P}^*_m \mid p$ has all its zeros on $\Gamma_1\}$	67
$\mathcal{A}^*_n(\mathbf{P}^*)$	$\lim\limits_{m\to\infty} \mathcal{A}_n(\mathbf{P}^*_m)$	68
$\mathcal{A}^*_n(\mathbf{P}^\Gamma)$	$\limsup\limits_{m\to\infty} \mathcal{A}_n(\mathbf{P}^\Gamma_m)$	68
$\delta\mathbf{P}^*_m$	$\left\{p \in \cup_{j=1}^m \mathbf{P}^*_j \mid 0 \in \overline{p(\overline{\Delta_1})}\right\}$	73
$\mathbf{P}^*_m(\mathbf{R})$	$\{f \in \mathbf{P}^*_m \mid f$ has real coefficients$\}$	75
$\delta\mathbf{P}^*_m(\mathbf{R})$	$\{f \in \delta\mathbf{P}^*_m \mid f$ has real coefficients$\}$	75

Chapter 1

Preliminaries

1.1 Introduction

Let $H(\Delta_1)$ denote the set of functions f, holomorphic in Δ_1. If $f \in H(\Delta_1)$, its Taylor expansion can be written as

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (|z| < 1),$$

where $a_n = \frac{f^{(n)}(0)}{n!}$.

If W is a subset of $H(\Delta_1)$, we define

$$\mathcal{A}_n(W) = \sup_{f \in W} \left| \frac{f^{(n)}(0)}{n!} \right| \quad (n = 0, 1, 2, \ldots).$$

$W \subset H(\Delta_1)$ is called *rotation-invariant* if from $f \in W$ it follows that for all $\kappa, \xi \in \Gamma_1$ the functions $f_{\kappa,\xi} : z \to \kappa f(\xi z)$ are also elements of W. Note that if W is a rotation-invariant subset of $H(\Delta_1)$, then

$$\mathcal{A}_n(W) = \sup_{f \in W, f(0) \geq 0, f^{(n)}(0) \geq 0} \frac{f^{(n)}(0)}{n!} \quad (n = 0, 1, 2, \ldots).$$

In this thesis we shall study $\mathcal{A}_n(W)$ for several subsets W of $H(\Delta_1)$. Most of them are closely related to the subset $\mathbf{B}^*$:

$$\mathbf{B}^* = \{f \in H(\Delta_1) \mid 0 < |f| \leq 1\}.$$

Note that $\mathbf{B}^*$ is rotation-invariant.

The problem to determine $\mathcal{A}_n(\mathbf{B}^*)$ was proposed by J. Krzyż [13], who also conjectured that

Conjecture (Krzyż): *For $n > 1$: $\mathcal{A}_n(\mathbf{B}^*) = \frac{2}{e}$.*

(Of course $\mathcal{A}_0(\mathbf{B}^*) = 1$). The conjectured value is attained by the functions

$$f_{\kappa,\xi} : z \to \kappa \exp\left(\frac{\xi z^n - 1}{\xi z^n + 1}\right) = \kappa\left(\frac{1}{e} + \frac{2}{e}\xi z^n - \frac{2}{3e}\xi z^{3n} + \dots\right) \quad (\kappa, \xi \in \Gamma_1)$$

and no others.

1.2 History and Conjectures

The Krzyż-conjecture for $n = 1, 2$ is not difficult to prove. Krzyż himself, amongst others, proved it. (In this thesis, it follows immediately from Theorem 2.5).

Hummel et al. [11] were the first to prove it for $n = 3$. Their proof is quite elaborate and makes use of a variational method. Prokhorov and Szynal [24] state that they are able to prove it using some estimates which were also deduced by Campschroer [4], pp. 34-37. We give a very elementary proof in Theorem 2.7.

Several authors (Brown [3], Delin Tan [5]) claim that they proved the Krzyż-conjecture for $n = 4$ but fail to give (full) details. We shall give a proof in Theorem 2.10. For larger n, the Krzyż-conjecture still remains unknown to be true.

Hummel et al. [11] also consider the subclass $\mathbf{B}^*_U \subset \mathbf{B}^*$ consisting of those elements of $\mathbf{B}^*$ which are univalent. They conjecture:

Conjecture (Hummel, Scheinberg and Zalcman): *For $n \geq 1$: A function $f \in \mathbf{B}^*_U$ which maximizes $\mathcal{A}_n(\mathbf{B}^*_U)$ maps Δ_1 onto Δ_1 slit radially from 0 to the boundary.*

We shall look at this class in depth in Chapter 3. The Hummel-conjecture in case $n = 1$ was first proved by MacGregor (unpublished). We prove it in Theorem 3.1.

Prokhorov and Szynal [24] determined $\mathcal{A}_2(\mathbf{B}^*_U)$ but their proof is far from elegant and heavily computer-based. Our proof (Theorem 3.2), again, is elementary.

Since the polynomial elements of $\mathbf{B}^*$ form a dense subclass of $\mathbf{B}^*$ (this will be shown in Theorem 4.1) it is natural to consider this subclass in depth. Saff and Sheil-Small [27] consider the subclass of those polynomials whose zeros lie on Γ_1. They conjecture:

Conjecture (Saff and Sheil-Small): *For $n \geq 1$:*

$$\mathcal{A}_n^*(\mathbf{P}^\Gamma) \leq C < \frac{1}{2}.$$

We prove this conjecture in Chapter 4 using a result due to Horowitz [10].

1.3 General Properties

1.3.1 The Topology on $H(\Delta_1)$

If $f, g \in H(\Delta_1)$ we define

$$d(f,g) = \sum_{k=1}^{\infty} d_k(f,g)2^{-k}$$

where

$$d_k(f,g) = \sup_{|z| \leq 1-\frac{1}{k}} \frac{|f(z)-g(z)|}{1+|f(z)-g(z)|}.$$

Then d is a metric on $H(\Delta_1)$.
We therefore can define a topology T on $H(\Delta_1)$, derived from this metric. It is easy to verify that convergence with respect to d is the same as locally uniform convergence. The topology T will be used throughout the sequel.

1.3.2 Extremal Functions for $\mathcal{A}_n(\mathbf{B}^*)$

Let $n \in \mathbf{N}$. Let W be a subset of $H(\Delta_1)$.

We call $f \in W$ *extremal* for $\mathcal{A}_n(W)$ if $f(z) = \sum_{j=0}^{\infty} a_j z^j$ and $\Re e\ a_n = \mathcal{A}_n(W)$.

At forehand it is not clear whether an extremal function for $\mathcal{A}_n(\mathbf{B}^*)$ exists, but fortunately this is the case.

Theorem 1.1 *For each $n \in \mathbf{N}$ there exists an extremal function f for $\mathcal{A}_n(\mathbf{B}^*)$.*

Proof:
(See Duren [6], pp. 9-10).
The set $F = \{f \mid f \in \mathbf{B}^* \text{ or } f \equiv 0\}$ is compact. Furthermore, the mapping

$$\begin{aligned} \phi_n : F &\rightarrow \mathbf{R} \\ f &\rightarrow \frac{\Re e\ f^{(n)}(0)}{n!} \end{aligned}$$

is continuous. Therefore ϕ_n attains its maximum at some point of F. Since $f \equiv 0$ certainly *does not* maximize ϕ_n, there exists $f \in \mathbf{B}^*, f(z) = \sum_{j=0}^{\infty} a_j z^j$, such that

$$\Re e\, a_n = \phi_n(f) = \max_{g \in \mathbf{B}^*} \phi_n(g) = \mathcal{A}_n(\mathbf{B}^*).$$

■

1.3.3 Subordination

Define $\mathbf{B}_0 \subset H(\Delta_1)$ by

$$\mathbf{B}_0 = \{w \in H(\Delta_1) \mid w(0) = 0, |w| < 1\}.$$

Suppose $f, g \in H(\Delta_1)$ and $f(0) = g(0)$. Furthermore, suppose that f is *univalent* (i.e. *injective*) and that $g(\Delta_1) \subseteq f(\Delta_1)$. Then

$$w : z \to f^{-1} \circ g(z) \quad (z \in \Delta_1)$$

is well-defined and is an element of $\mathbf{B}_0$. From Schwarz' lemma it follows that $|w(z)| \leq |z|$ for all $z \in \Delta_1$ and this implies that $g(\Delta_r) \subseteq f(\Delta_r)$. This is the so-called subordination principle. In a slightly more general setting:

Definition: *Let $f, g \in H(\Delta_1), f(0) = g(0)$. g is said to be subordinate to f, if there exists a $w \in \mathbf{B}_0$ such that $g = f \circ w$.*

If we know that g is subordinate to f, we can derive many properties of g from those of f. For example, we have just seen that for each $r \in (0, 1)$:

$$g(\Delta_r) \subseteq f(\Delta_r).$$

From this we immediately find:

- $|g'(0)| \leq |f'(0)|$,
- $M_\infty(r, g) \leq M_\infty(r, f)$.

Littlewood [16] showed that the result stated above can be extended to

Theorem 1.2 *Let $f, g \in H(\Delta_1)$, let g be subordinate to f. Then for all $p \in (0, \infty)$:*

$$M_p(r, g) \leq M_p(r, f) \quad (r \in (0, 1)).$$

Strict inequality holds for all r unless f is constant or g is a rotation of f (i.e. $g(z) = f(\kappa z)$, $|\kappa| = 1$, $z \in \Delta_1$).

Proof:
See Duren [6], pp. 191-192.

■

Taking $p = 2$ in Theorem 1.2 we find:

Corollary 1.1 *If $f, g \in H(\Delta_1)$ and if g is subordinate to f, $f(z) = \sum_{k=1}^{\infty} a_k z^k, g(z) = \sum_{k=1}^{\infty} b_k z^k$, then*

$$\sum_{k=1}^{\infty} |b_k|^2 \leq \sum_{k=1}^{\infty} |a_k|^2.$$

■

Rogosinski [25] proved the stronger results:

Theorem 1.3 (Rogosinski) *If $f, g \in H(\Delta_1)$ and if g is subordinate to f, $f(z) = \sum_{k=1}^{\infty} a_k z^k, g(z) = \sum_{k=1}^{\infty} b_k z^k$, then for all $n \in \mathbf{N}$:*

$$\sum_{k=1}^{n} |b_k|^2 \leq \sum_{k=1}^{n} |a_k|^2.$$

Moreover, if both f and g are univalent then for all $n \in \mathbf{N}$:

$$\sum_{k=1}^{n} k|b_k|^2 \leq \sum_{k=1}^{n} k|a_k|^2.$$

■

1.3.4 The Class P

Define $\mathbf{P} \subset H(\Delta_1)$ by

$$\mathbf{P} = \{f \in H(\Delta_1) \mid f(0) = 1, \ \Re e\, f(z) > 0 \quad (z \in \Delta_1)\}.$$

For example, $f : z \to \frac{1+z}{1-z} \in \mathbf{P}$.
(Note that $\frac{1+z}{1-z} = 1 + \sum_{j=1}^{\infty} 2z^j$). This function indeed is an important member of the class $\mathbf{P}$ which can be seen from the following theorem:

Theorem 1.4 *Let $p \in \mathbf{P}$. Then p is subordinate to $f : z \to \frac{1+z}{1-z}$.*

Proof:
Let $p \in \mathbf{P}$. Define $w : z \to \frac{p(z)-1}{p(z)+1}$. Then $w \in \mathbf{B}_0$ and $p = f \circ w$.

■

Corollary 1.2 *If $p \in \mathbf{P}$ then*

- $\frac{1-|z|}{1+|z|} \le |p(z)| \le \frac{1+|z|}{1-|z|} \quad (z \in \Delta_1)$,
- $|p'(0)| \le 2$.

■

Let $f \in \mathbf{B}^*, f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Since $f \neq 0$ in Δ_1 and Δ_1 is simply connected, there exists a function $g \in H(\Delta_1)$ such that $f = \exp(g)$. Since $|f| < 1$ in Δ_1, there exist $p \in \mathbf{P}, \alpha > 0, \kappa \in \Gamma_1$ such that $f = \kappa \cdot \exp(-\alpha p)$.

Theorem 1.5 *Let $g \in \mathbf{B}^*$. Then g is subordinate to $f : z \to \kappa \cdot \exp\left(-\alpha \frac{1+z}{1-z}\right)$ with $\alpha = -\log|g(0)|$, $\kappa = \frac{g(0)}{|g(0)|}$.*

Proof:
Trivial.

■

Corollary 1.3 $|g'(0)| \le 2\alpha e^{-\alpha}$ *where* $\alpha = -\log|g(0)|$.

■

From this we see that coefficient problems for the class $\mathbf{B}^*$ are closely related to those for the class $\mathbf{P}$.

Theorem 1.6 (Herglotz Representation) *Let $p \in \mathbf{P}$. Then there exists a positive unit measure μ on $[0, 2\pi)$ such that*

$$p(z) = \int_0^{2\pi} \frac{z + e^{it}}{z - e^{it}} d\mu(t) \quad (|z| < 1). \tag{1.1}$$

Proof:
See Herglotz [9], Pommerenke [23].

■

Note: On the other hand, if μ is a positive unit measure on $[0, 2\pi)$ it is easy to verify that a function defined as in (1.1) is an element of $\mathbf{P}$.

Theorem 1.7 (Caratheodory) *Let $p \in \mathbf{P}$, $p(z) = 1 + \sum_{j=1}^{\infty} c_j z^j$.*

Then

$$|c_j| \le 2, \quad \forall j \in \mathbf{N}.$$

If, for any j, $|c_j| = 2$, then p is of the form

$$p(z) = \sum_{k=1}^{j} \lambda_k \frac{e^{i\alpha + \frac{2\pi i k}{j}} + z}{e^{i\alpha + \frac{2\pi i k}{j}} - z} \tag{1.2}$$

for some $\alpha \in [0, 2\pi]$ and $\lambda_1, \lambda_2, ..., \lambda_j \geq 0$ with $\sum_{k=1}^{j} \lambda_k = 1$.

Proof:
Write $p(z) = \int_0^{2\pi} \frac{z + e^{it}}{z - e^{it}} d\mu(t), \quad (|z| < 1)$. Then

$$c_j = 2 \int_0^{2\pi} e^{-ijt} d\mu(t).$$

Therefore

$$|c_j| \leq 2 \int_0^{2\pi} d\mu(t) = 2.$$

We have equality if and only if e^{-ijt} is constant on the support of μ. Therefore, μ must be a sum of point measures whose support is at points of the form $\alpha + \frac{2\pi l}{j}$. From this the theorem follows.

■

Theorem 1.8 *Let $p \in H(\Delta_1)$, $p(z) = 1 + \sum_{j=1}^{\infty} c_j z^j$. Define*

$$\begin{aligned} c_j &= \overline{c_{-j}} \quad (j < 0), \\ c_0 &= 2. \end{aligned}$$

Then $p \in \mathbf{P}$ if and only if for all $n \in \mathbf{N}$ and all $\lambda_0, \lambda_1, ..., \lambda_n \in \mathbf{C}$

$$\sum_{k,l=0}^{n} c_{k-l} \lambda_k \overline{\lambda_l} \geq 0.$$

Proof:
Suppose $p \in \mathbf{P}$.
Then according to Theorem 1.6, there exists a positive measure μ such that

$$c_j = 2 \int_0^{2\pi} e^{-ijt} d\mu(t) \quad (j \in \mathbf{Z}(!))$$

Therefore, for all $n \in \mathbf{N}$ and $\lambda_0, \lambda_1, ..., \lambda_n \in \mathbf{C}$:

$$\int_0^{2\pi}\left|\sum_{k=0}^{n}\lambda_k e^{-ikt}\right|^2 d\mu(t) = \int_0^{2\pi}\left(\sum_{k=0}^{n}\lambda_k e^{-ikt}\sum_{l=0}^{n}\overline{\lambda_l}e^{ilt}\right)d\mu(t) \quad (1.3)$$

$$= \sum_{k=0}^{n}\sum_{l=0}^{n}\lambda_k\overline{\lambda_l}\int_0^{2\pi} e^{-i(k-l)t}d\mu(t)$$

$$= \frac{1}{2}\sum_{k,l=0}^{n} c_{k-l}\lambda_k\overline{\lambda_l}. \quad (1.4)$$

Since μ is a positive measure the L.H.S. of (1.3) (and therefore the R.H.S. of (1.4)) is non-negative.

Now suppose that for all $n \in \mathbf{N}$, and all $\lambda_0, \lambda_1, ..., \lambda_n \in \mathbf{C}$ we have:

$$\sum_{k,l=0}^{n} c_{k-l}\lambda_k\overline{\lambda_l} \geq 0.$$

Then, by taking $\lambda_k = z^k \quad (k \in \mathbf{N})$ and $n \to \infty$, we see that for all $|z| < 1$:

$$\begin{aligned}
0 &\leq \sum_{k,l=0}^{\infty} c_{k-l}z^k\overline{z}^l \\
&= 2\sum_{k=0}^{\infty}|z|^{2k} + 2\Re e\sum_{k=0}^{\infty}\sum_{l=k+1}^{\infty} c_{k-l}|z|^{2k}\overline{z}^{l-k} \\
&= 2\sum_{k=0}^{\infty}|z|^{2k} + 2\Re e\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}c_{-j}\overline{z}^j|z|^{2k} \\
&= \frac{2}{1-|z|^2} + \frac{2}{1-|z|^2}\Re e\sum_{j=1}^{\infty}\overline{c_j z^j} \\
&= \frac{2}{1-|z|^2}\Re e\, p(z).
\end{aligned}$$

Therefore $\Re e\ p(z) \geq 0 \quad (z \in \Delta_1)$. Since $p(0) = 1$, it follows from the maximum modulus principle that $\Re e\, p(z) > 0$, $(z \in \Delta_1)$, i.e. $p \in \mathbf{P}$.

■

1.3.5 The Class B

Define $\mathbf{B} \subset H(\Delta_1)$ by

$$\mathbf{B} = \{f \in H(\Delta_1) \mid |f(z)| \leq 1 \quad (z \in \Delta_1)\}.$$

For example $f : z \to z \in \mathbf{B}$.
Moreover, for every $a \in \Delta_1$, $f_a : z \to \frac{z-a}{1-\overline{a}z} \in \mathbf{B}$.

As with the classes **P** and **B***, subordination theorems can be successfully applied to the class **B**:

Theorem 1.9 *Let* $g \in \mathbf{B}, a = g(0)$.
Then g *is subordinate to* $f_{-a} : z \to \frac{z+a}{1+\bar{a}z}$.

Proof:
Define $w : z \to \frac{g(z)-a}{1-\bar{a}g(z)}$.
Then $w \in \mathbf{B}_0$ and $g = f_{-a} \circ w$.

■

Corollary 1.4 *Let* $g \in \mathbf{B}, a = g(0)$. *Then:*

- $|g'(0)| \leq 1 - |a|^2$ *(Pick's Lemma),*
- $\max\left(0, \frac{|a|-|z|}{1-|a||z|}\right) \leq |g(z)| \leq \frac{|a|+|z|}{1+|a||z|}$.

Corollary 1.5 *Let* $g \in \mathbf{B}, g(z) = \sum_{j=0}^{\infty} a_j z^j$. *Then:*

$$\sum_{j=0}^{n} |a_j|^2 \leq 1 - (1 - |a_0|^2)|a_0|^{2n}, \quad \forall n \in \mathbf{N}.$$

■

(For $n = 1$, this is Pick's Lemma again).

Theorem 1.10 *Let* $f \in H(\Delta_1), f(z) = \sum_{j=0}^{\infty} a_j z^j$. *Define* $s_n(z) = \sum_{j=0}^{n} a_j z^j$.
Then the following are equivalent:

1. $f \in \mathbf{B}$.
2. $\left|\sum_{j=0}^{n} s_j(z)\right| \leq n+1, \quad (\forall z \in \Delta_1, \forall n \in \mathbf{N})$.
3. $\sum_{j=0}^{n} |s_j(z)| \leq n+1, \quad (\forall z \in \Delta_1, \forall n \in \mathbf{N})$.
4. $\sum_{j=0}^{n} |s_j(z)|^2 \leq n+1, \quad (\forall z \in \Delta_1, \forall n \in \mathbf{N})$.

Proof:
See Landau [14], pp. 22-25.

■

Note that, if $f \in \mathbf{B}$, then $g : z \to zf(z)$ is an element of $\mathbf{B}_0$. On the other hand, if $f \in \mathbf{B}_0$, then $g : z \to \frac{f(z)}{z}$ is an element of $\mathbf{B}$.

We have already seen a correspondence between the classes $\mathbf{B}^*$ and $\mathbf{P}$. The correspondence between $\mathbf{P}$ and $\mathbf{B}_0$ is obvious:

$$f \in \mathbf{B}_0 \Leftrightarrow \frac{1-f}{1+f} \in \mathbf{P}.$$

From this:

$$f \in \mathbf{B}^* \Leftrightarrow f = \lambda \exp\left(\alpha \cdot \frac{w-1}{w+1}\right), \quad |\lambda| = 1,\ \alpha \geq 0,\ w \in \mathbf{B}_0.$$

1.3.6 The Class S

Define $\mathbf{S} \subset H(\Delta_1)$ by

$$\mathbf{S} = \{f \in H(\Delta_1) \mid f(0) = 0, f'(0) = 1, \text{ f is injective}\}.$$

For example $f : z \to \frac{z}{(1-z)^2} \in \mathbf{S}$.
Note: An analytic, injective function is usually called *univalent* or *schlicht*.

The class $\mathbf{S}$ is a very famous subclass of $H(\Delta_1)$. Many books on the class $\mathbf{S}$ have been written, see e.g. Duren [6], Pommerenke [23].

The function f mentioned above is called the *Koebe* function and has the property that $f(z) = \sum_{n=1}^{\infty} nz^n$. Bieberbach [1] conjectured:

Conjecture (Bieberbach): Let $f \in \mathbf{S}$, $f(z) = \sum_{n=1}^{\infty} a_n z^n$. Then for all $n \in \mathbf{N}$:

$$|a_n| \leq n.$$

He showed that this is true for $n = 2$ (1916).

Since then, the conjecture has attracted many mathematicians.
Loewner was the first to introduce an important tool, the now so-called Loewner differential equation:

Theorem 1.11 *To each single-slit mapping f there exists a family of mappings $f(\ ,t)$ $(t \in [0,\infty))$ such that for every $t \in [0,\infty)$, $f(\ ,t)$ is an element of* $\mathbf{B}$ *and such that*

- *f satisfies the differential equation*

$$\frac{\partial}{\partial t} f = -f \frac{1+\kappa(t)f}{1-\kappa(t)f}$$

where $\kappa(t)$ is a continuous function with $|\kappa(t)| = 1$,

- $f(z,0) = z$,
- $\lim_{t\to\infty} e^t f(z,t) = f(z) \quad (|z| < 1)$, *uniformly on compact sets.*

Since the single-slit mappings form a dense subclass of **S** (see Duren [6], pp. 80-81) it follows that we may restrict ourselves to functions generated by the differential equation given above.

With the aid of his equation, Loewner was able to prove the Bieberbach Conjecture for $n = 3$ (1923) (see Loewner [19]).

To follow history, Garabedian and Schiffer [8] proved it for $n = 4$ (1955), Pederson [20] for $n = 6$ (1968) and Pederson and Schiffer [21] for $n = 5$ (1972).

Finally, in 1984 de Branges [2] astonished the whole mathematical world by proving it *for all* n, using the Loewner differential equation and Hilbert space techniques.

1.4 Schur's Method

There exists a very elegant way to derive coefficient (in-)equalities for functions in **B**. (And from these, (in-)equalities will follow for the classes $\mathbf{B}^*$ and **P** as well). The method goes back to Schur [28] and has later been revised by several authors (see Garnett [7], Campschroer [4], pp. 16-18).

Let $n \in \mathbf{N}$ and let $Gl(n)$ denote the space of $n \times n$-matrices. Define

$$\begin{aligned} T_n : H(\Delta_1) &\rightarrow Gl(n) \\ (T_n(f))_{ij} &= \begin{cases} 0 & j < i \\ a_{j-i} & j \geq i \end{cases} \end{aligned}$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^k$.

Then T_n is a lineair mapping from $H(\Delta_1)$ into $Gl(n)$. Moreover, T_n is multiplicative, i.e. $T_n(fg) = T_n(f)T_n(g)$ for all $f, g \in H(\Delta_1)$ (see Campschroer [4], p. 17).

In $\mathbf{C}^n$ we have the euclidean norm $\|\ \|$:

$$\|z\|^2 = \langle z, z\rangle = \sum_{i=1}^{n} |z_i|^2$$

where $z = (z_1, z_2, ..., z_n)$.

(Here $\langle\ ,\ \rangle$ denotes the usual inner product on $\mathbf{C}^n$). With the aid of this norm we will compare matrices, and therefore elements of $H(\Delta_1)$ with each other.

Definition: *Let $A, B \in Gl(n)$. We say that B majorizes A (in Schur's sense) if for all $z \in \mathbf{C}^n$: $\|Az\| \leq \|Bz\|$. Notation: $A \ll B$.*

Definition: *Let $f, g \in H(\Delta_1)$. We say that g majorizes f (in Schur's sense) if for all $n \in \mathbf{N}$: $T_n(f) \ll T_n(g)$. Notation: $f \ll g$.*

The following theorem can easily be proved:

Theorem 1.12 (Campschroer) *Let $A, B, C \in Gl(n)$. If $A \ll B$ then*

1. *$AC \ll BC$,*
2. *If $C \ll I_n$ then $CA \ll B$ (I_n is the identity-matrix on C^n),*
3. *for all $z \in \overline{\Delta_1}$ we have $A - zB \ll B - \overline{z}A$,*
4. *If $A = \begin{pmatrix} \emptyset & A_1 \\ 0 & \emptyset \end{pmatrix}$ and $B = \begin{pmatrix} x & w \\ 0 & B_1 \end{pmatrix}$, ($w \in \mathbf{C}^{n-1}$, $x \neq 0$, $A_1, B_1 \in Gl(n-1)$), then $A_1 \ll B_1$.*

■

We can rewrite the result of Theorem 1.8 as follows:

Theorem 1.13 *Let $p \in H(\Delta_1)$, $p(0) = 1$. Then $p \in \mathbf{P}$ if and only if for all $n \in \mathbf{N}$ and all $y \in \mathbf{C}^n$:*

$$\langle (T_n(p) + T_n^*(p))y, y \rangle \geq 0$$

(Here T_n^* denotes the transposed matrix of T_n).

■

We now prove a corresponding form of the previous theorem for elements in $\mathbf{B}$:

Theorem 1.14 *Let $w \in H(\Delta_1)$. Then $w \in \mathbf{B}$ if and only if $w \ll \mathbf{1}_{\Delta_1}$.*

Proof:
First suppose $w \in \mathbf{B}, n \in \mathbf{N}$.
Let $p : z \to \frac{1+zw(z)}{1-zw(z)} \in \mathbf{P}$. Then

$$\langle (T_{n+1}(p) + T_{n+1}^*(p))y, y \rangle \geq 0 \quad (y \in \mathbf{C}^{n+1}).$$

Let $x \in \mathbf{C}^{n+1}, y = T_{n+1}(1 - zw(z))x,\ S_{n+1} = T_{n+1}(zw(z))$. Then

$$
\begin{aligned}
& \langle T_{n+1}(p)T_{n+1}(1 - zw(z))x, T_{n+1}(1 - zw(z))x\rangle + \\
& \langle T_{n+1}(1 - zw(z))x, T_{n+1}(p)T_{n+1}(1 - zw(z))x\rangle \geq 0 \\
\Rightarrow \quad & \langle T_{n+1}(1 + zw(z))x, T_{n+1}(1 - zw(z))x\rangle + \\
& \langle T_{n+1}(1 - zw(z))x, T_{n+1}(1 + zw(z))x\rangle \geq 0 \\
\Rightarrow \quad & \langle (I_{n+1} + S_{n+1})x, (I_{n+1} - S_{n+1})x\rangle + \\
& \langle (I_{n+1} - S_{n+1})x, (I_{n+1} + S_{n+1})x\rangle \geq 0 \\
\Rightarrow \quad & \langle x, x\rangle \geq \langle S_{n+1}x, S_{n+1}x\rangle \\
\Rightarrow \quad & T_{n+1}(zw(z)) \ll I_{n+1}.
\end{aligned}
$$

From Theorem 1.12 it follows that $T_n(w) \ll I_n$ and from this that $w \ll \mathbf{1}_{\Delta_1}$.

If, on the other hand, we have for all $n \in \mathbf{N}$, $T_n(w) \ll I_n$, then, by considering the vector $\vec{z} = (1, z, z^2, ..., z^{n-1})$, we find that for all $n \in \mathbf{N}$ and all $z \in \Gamma_1$:

$$\langle (T_n(w))\vec{z}, (T_n(w))\vec{z}\rangle \ \leq \langle \vec{z}, \vec{z}\rangle$$

i.e.

$$\sum_{j=0}^{n-1} |s_j(z)|^2 \leq n.$$

From Theorem 1.10 we conclude that $w \in \mathbf{B}$.

■

Theorem 1.15 *Let $p \in \mathbf{P}, q : z \to \frac{p(z)-1}{z} \in H(\Delta_1)$. Then $q \ll p + 1$.*

Proof:
Let $w : z \to \frac{1}{z}\frac{p(z)-1}{p(z)+1}$ and $n \in \mathbf{N}$. Then $w \in \mathbf{B}$ and therefore $T_n(w) \ll I_n$. Then $T_n(w)T_n(p+1) \ll T_n(p+1)$ and, since T_n is multiplicative,

$$T_n(q) \ll T_n(p+1).$$

■

Example 1.1

If $w \in \mathbf{B}, w(z) = \sum_{j=0}^{\infty} a_j z^j$, we have $w \ll \mathbf{1}_{\Delta_1}$.
In particular we find $T_2(w) \ll T_2(\mathbf{1}_{\Delta_1})$, i.e.

$$\begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix} \ll \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now consider $\vec{x} = (\frac{a_1\overline{a_0}}{1-|a_0|^2}, 1)$. A brief calculation shows that $|a_1| \leq 1 - |a_0|^2$ (Pick's lemma, again).

Example 1.2

If $p \in \mathbf{P}, p(z) = 1 + \sum_{j=1}^{\infty} c_j z^j$, we have $\frac{p-1}{z} \ll p+1$. From this we find:

$$\begin{pmatrix} c_1 & c_2 \\ 0 & c_1 \end{pmatrix} \ll \begin{pmatrix} 2 & c_1 \\ 0 & 2 \end{pmatrix}.$$

Therefore:

$$|c_2 + \lambda c_1^2|^2 \leq 4 + 4|c_1|^2(|\lambda|^2 + \Re e\, \lambda) \quad (\lambda \in \mathbf{C}).$$

Furthermore

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ & c_1 & c_2 \\ \emptyset & & c_1 \end{pmatrix} \ll \begin{pmatrix} 2 & c_1 & c_2 \\ & 2 & c_1 \\ \emptyset & & 2 \end{pmatrix}.$$

Consider the vectors $\vec{x}_1 = (0, 0, 1)$,

$\vec{x}_2 = (0, -c_1, 1)$,

$\vec{x}_3 = (c_1^2 - c_2, -c_1, 1)$.

We find $|c_3| \leq 2$,

$|c_3 - c_1 c_2| \leq 2$,

$|c_3 - 2c_1 c_2 + c_1^3| \leq 2$.

(See Livingston [17], Campschroer [4], p. 18).

Chapter 2

Coefficient Estimates in $\mathbf{B}^*$

2.1 Introduction

In this chapter we shall take a closer look at the class $\mathbf{B}^*$. Our main goal is to prove the Krzyż-conjecture for the first four coefficients, i.e.

Theorem 2.1 *Let $n \in \{1, 2, 3, 4\}$ and $f \in \mathbf{B}^*, f(z) = \sum_{j=0}^{\infty} a_j z^j$. Then*

$$|a_n| \le \frac{2}{e}.$$

Equality is attained only for $: z \to f_{\kappa,\xi}(z) = \kappa \exp\left(\frac{\xi z^n - 1}{\xi z^n + 1}\right) \quad (\kappa, \xi \in \Gamma_1)$.

Furthermore, we shall give an upper bound for $\{\mathcal{A}_n(\mathbf{B}^*) \mid n \in \mathbf{N}\}$ and give applications of the techniques introduced in the previous chapter: In section 2.5 we give sharp estimates for coefficients of functions f which map Δ_1 into a ring-shaped domain, i.e.

$$r < |f(z)| < s \quad (z \in \Delta_1)$$

where $0 < r < s < \infty$.

2.2 Variational Methods in the Class $\mathbf{B}^*$

Let $n \in \mathbf{N}$. If we are looking for an extremal function for $\mathcal{A}_n(\mathbf{B}^*)$ we may apply so-called *variational* methods to obtain properties of extremal functions. The strategy is to perturbate a given function in order to derive whether that function *can* be extremal, or, which properties an extremal function must have.

Example 2.1 (Marty variation) *Let $n \in \mathbf{N}$. Suppose $f \in \mathbf{B}^*$ is extremal for $\mathcal{A}_n(\mathbf{B}^*)$, $f(z) = \sum_{j=0}^{\infty} a_j z^j$. Then*

$$(n+1)a_{n+1} = (n-1)\overline{a_{n-1}}.$$

Proof:
If $f \in \mathbf{B}^*$ is extremal and $\xi \in \Delta_1$, the function

$$f_\xi : z \to f\left(\frac{z+\xi}{1+\overline{\xi}z}\right)$$

is also in $\mathbf{B}^*$. Now, f is extremal, therefore we find:

$$\Re e f_\xi^{(n)}(0) \leq \Re e f^{(n)}(0).$$

We express the n^{th} coefficient of f_ξ in terms of the coefficients of f and ξ:
Note that

$$\left(\frac{z+\xi}{1+\overline{\xi}z}\right)^m = z^m + m(\xi - \overline{\xi}z^2)z^{m-1} + \mathcal{O}(|\xi|^2).$$

Hence

$$f_\xi(z) = f\left(\frac{z+\xi}{1+\overline{\xi}z}\right) = \sum_{m=1}^{\infty} a_m[z^m + m(\xi - \overline{\xi}z^2)z^{m-1}] + \mathcal{O}(|\xi|^2).$$

The n^{th} coefficient of $f_\xi(z)$ is

$$a_n + (n+1)a_{n+1}\xi - (n-1)a_{n-1}\overline{\xi} + \mathcal{O}(|\xi|^2).$$

Thus:

$$\Re e \ \left(\xi[(n+1)a_{n+1} - (n-1)\overline{a_{n-1}}]\right) + \mathcal{O}(|\xi|^2) \leq 0 \quad (\xi \in \Delta_1).$$

The result now follows by letting $\xi \to 0$.

■

Of course, the conjectured extremal function $f : z \to \exp(-\frac{1-z^n}{1+z^n})$ satisfies the Marty relation. Unfortunately, there are many more functions in $\mathbf{B}^*$, which satisfy this relation (for example, if n is odd, every even function trivially satisfies it). Although we shall not use the Marty relation in the sequel, the method of variation as described above can be put to good use, as can be seen from the following theorem.

This theorem states, more or less, that, if we are looking for extremal functions, we can restrict ourselves to a (very small) subclass of $\mathbf{B}^*$. The result was first proved by Hummel et al. [11] using a method due to Goluzin. However the following proof is more elegant (and simpler). It was pointed out by R. Kortram:

Theorem 2.2 *Let $n \in \mathbf{N}$. The extremal functions of $\mathcal{A}_n(\mathbf{B}^*)$ are of the form:*

$$f(z) = \exp\left(-\sum_{k=1}^{n} \lambda_k \frac{1+e^{i\theta_k}z}{1-e^{i\theta_k}z}\right)$$

where $\lambda_k \geq 0$, $\theta_k \in [0, 2\pi)$, $k = 1, 2, ..., n$.

Proof:
Let

$$\begin{array}{rccl} L: & H(\Delta_1) & \to & \mathbf{R} \\ & f & \to & \Re e\, \frac{f^{(n)}(0)}{n!}. \end{array}$$

Then L is a lineair functional. Furthermore, let $k_\xi : z \to \frac{1+\xi z}{1-\xi z}$ and

$$\begin{array}{rccl} \phi_f: & \Gamma_1 & \to & \mathbf{R} \\ & \xi & \to & L(fk_\xi) \end{array}$$

i.e.

$$\phi_f(\xi) = \Re e\,(a_n + 2a_{n-1}\xi + ... + 2a_0\xi^n).$$

On Γ_1, ϕ_f vanishes at most $2n$ times. Suppose $f \in \mathbf{B}^*$ is an extremal function for $L_{|\mathbf{B}^*}$. Let $t > 0$, $\xi \in \overline{\Delta_1}$ and define $f_{t,\xi} : z \to f(z)\exp(-tk_\xi(z))$. Then $f_{t,\xi} \in \mathbf{B}^*$ and therefore $L(f_{t,\xi}) \leq L(f)$. Since for small $t > 0$, $f_{t,\xi}$ is a small perturbation of f, $L(f_{t,\xi})$ will be a small perturbation of $L(f)$. More precisely:

$$L(f_{t,\xi}) = L(f) - tL(fk_\xi) + \mathcal{O}(t^2).$$

Therefore:

$$\phi_f(\xi) \geq 0 \quad (\xi \in \Gamma_1).$$

We shall need a second variation:
From Herglotz' Representation we find a positive measure μ on $[0, 2\pi)$ such that $f(z) = \exp\left(-\int_0^{2\pi} \frac{z+e^{i\theta}}{z-e^{i\theta}} d\mu(\theta)\right)$. Now, let $A \subseteq [0, 2\pi)$ be measurable and define

$$\nu = \mu_{|A}.$$

For $-1 < t < 1$, $\mu + t\nu$ is a positive measure, and therefore

$$f_A : z \to f(z)\exp\left(-t\int_0^{2\pi} k_{e^{-i\theta}}(z)d\nu(\theta)\right) \in \mathbf{B}^*$$

i.e.

$$f_A : z \to f(z)\exp\left(-t\int_A k_{e^{-i\theta}}(z)d\mu(\theta)\right) \in \mathbf{B}^*.$$

From this we find for all measurable $A \subseteq [0, 2\pi)$:

$$L\left(f(z)\int_A k_{e^{-i\theta}}(z)d\mu(\theta)\right) = 0$$

i.e.

$$\int_A L\left(fk_{e^{-i\theta}}\right) d\mu(\theta) = 0. \tag{2.1}$$

Since (2.1) must hold for all measurable A it follows that μ must be a sum of point measures with weight only in those points θ where

$$\phi_f(e^{-i\theta}) = 0.$$

We have seen that $\phi_f(\xi)$ has at most $2n$ zeros on Γ_1. Since

$$\phi_f(e^{i\theta}) \geq 0 \quad (\theta \in [0, 2\pi)),$$

it follows that every zero must have even multiplicity and therefore μ is the sum of at most n point measures. From this the theorem follows.

■

Let $L : H(\Delta_1) \to \mathbf{C}$ be a lineair functional, $\Re e\ L_{|\mathbf{B}^*} \not\equiv 0$. Suppose that f is extremal for $\Re e\ L_{|\mathbf{B}^*}$ and that the mapping

$$\begin{aligned} \phi_f : \Delta_1 &\to \mathbf{C} \\ \xi &\to L(fk_\xi) \end{aligned}$$

is analytic. Then, since f is extremal for $\Re e\ L$,

$$\Re e\ L(fk_\xi) \geq 0 \quad (\xi \in \Delta_1),$$

i.e.

$$\Re e\ \phi_f(\xi) \geq 0.$$

From the maximum modulus principle it then follows that

$$\frac{\phi_f - \Im m\ \phi_f(0)}{\Re e\ \phi_f(0)} \in \mathbf{P}.$$

For example, if $L(f) = f^{(n)}(0)$, we find that

$$g : \xi \to a_n + 2a_{n-1}\xi + \ldots + 2a_0\xi^n$$

has positive real part in Δ_1. Therefore $|a_j| \leq |a_n| \quad (j = 0, 1, \ldots, n)$.

The following variation will improve the foregoing in case $j < \frac{n}{2}$:

Theorem 2.3 *Let $n \in \mathbf{N}$. If $f \in \mathbf{B}^*$, $f(z) = \sum_{j=0}^{\infty} a_j z^j$, is extremal for $\mathcal{A}_n(\mathbf{B}^*)$, then*

$$|a_j| \le \frac{1}{2} a_n \quad (j < \frac{n}{2}).$$

Proof:
For $t > 0$, $j < \frac{n}{2}$, $\theta \in [0, 2\pi)$, define

$$f_{t,j,\theta} : z \to f(z) \exp\left(-t \frac{1 + z^{n-j} e^{i\theta}}{1 - z^{n-j} e^{i\theta}}\right).$$

Then $f_{t,j,\theta} \in \mathbf{B}^*$. Its n^{th} coefficient is equal to

$$a_n - t(a_n + 2a_j e^{i\theta}) + \mathcal{O}(t^2).$$

Therefore

$$\Re e\,(a_n + 2a_j e^{i\theta}) \ge 0 \quad (\theta \in [0, 2\pi))$$

and from this $|a_j| \le \frac{1}{2} a_n$.

■

2.3 A Uniform Upper Bound for $\{\mathcal{A}_n(\mathbf{B}^*) | n \in \mathbf{N}\}$

The Krzyż-conjecture states that for all $n \in \mathbf{N}$, $\mathcal{A}_n(\mathbf{B}^*) = \frac{2}{e}$. We can not prove it for $n > 4$ and can (at forehand) only state a trivial estimate for

$$C(\mathbf{B}^*) = \sup_{n \in \mathbf{N}} \mathcal{A}_n(\mathbf{B}^*),$$

namely, $C(\mathbf{B}^*) \le 1$.

Horowitz [10] was the first to prove a non-trivial estimate:

$$C(\mathbf{B}^*) \le 1 - \frac{1}{3\pi} + \frac{4}{\pi} \sin\left(\frac{1}{12}\right).$$

In this section we give a simplified version of his proof and we obtain the sharper upper bound

$$C(\mathbf{B}^*) \le 1 - \frac{1}{5} + \frac{4}{\pi} \sin\left(\frac{\pi}{20}\right).$$

Although this isn't very stunning, compared to the conjectured value $\frac{2}{e}$, it is the best result as far as we know (see the note at the end of the proof).

Theorem 2.4 *For all $n \in \mathbf{N}$*

$$\mathcal{A}_n(\mathbf{B}^*) \le \frac{4}{5} + \frac{4}{\pi} \sin\left(\frac{\pi}{20}\right).$$

Before proving the theorem we derive three lemmata concerning integrals of the form $\int_a^b \cos(f(\theta))d\theta$.

Lemma 2.1
Suppose $f \in C^1[a,b]$, $\cos(f(\theta)) \geq 0, f'(\theta) \geq M > 0$ $(\theta \in (a,b))$.
Then

$$\left|\int_a^b \cos(f(\theta))d\theta\right| \leq \frac{2}{M}\sin\left(\frac{(b-a)M}{2}\right).$$

Proof:
(See Figure 2.1). We may assume that $f : [a,b] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$.

1. Suppose $f(c) = 0$, for some $c \in [a,b]$. Define

$$g : \theta \to M(\theta - c).$$

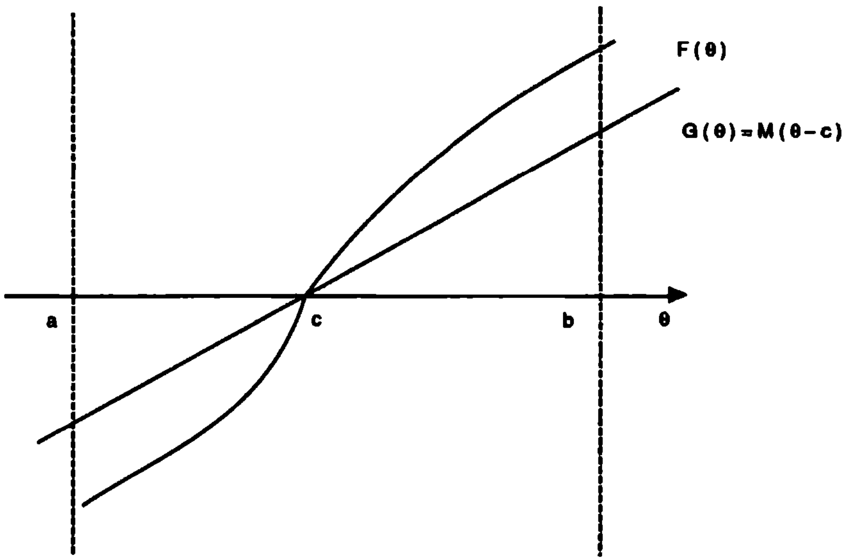

Figure 2.1

Consider

$$\int_a^b \cos(f(\theta)) - \cos(g(\theta))d\theta =$$

$$= -2 \quad \int_a^b \sin\left(\frac{f(\theta)+g(\theta)}{2}\right)\sin\left(\frac{f(\theta)-g(\theta)}{2}\right)d\theta \leq 0.$$

Thus

$$\left|\int_a^b \cos(f(\theta))d\theta\right| \leq \left|\int_a^b \cos(g(\theta))d\theta\right| = \frac{2}{M}\sin\left(\frac{(b-a)M}{2}\right).$$

2. If $f(\theta) > 0$ or $f(\theta) < 0$ throughout $[a,b]$ the above proof must be slightly modified.
(Consider $g : \theta \to M(\theta - a) + f(a)$ or $g : \theta \to M(\theta - b) + f(b)$).

■

Lemma 2.2
Suppose $f \in C^1[a,b]$, $f'(\theta) \geq M > 0$, $f''(\theta) \geq 0$ $(\theta \in (a,b))$.
Then

$$\left|\int_a^b \cos(f(\theta))d\theta\right| \leq \begin{cases} \frac{\sqrt{3}}{M} + \frac{b-a}{2} - \frac{1}{3}\pi M^{-1} & b - a \geq \frac{2}{3}\pi M^{-1} \\ \frac{2}{M}\sin\left(\frac{(b-a)M}{2}\right) & b - a < \frac{2}{3}\pi M^{-1}. \end{cases}$$

Proof:
Let $a \leq t_1 < t_2 \ldots < t_m \leq b$ denote the zeros of $\cos(f(\theta))$, then we have

$$f(t_j) = C + j \cdot \pi, \quad \text{where } C - \frac{\pi}{2} \in \pi\mathbf{Z}.$$

Note that $|t_{k+1} - t_k| = \frac{\pi}{f'(\xi_k)}$ for some $\xi_k \in (t_k, t_{k+1})$. Since $f'(\theta)$ is increasing this implies that

$$t_{k+1} - t_k \leq t_k - t_{k-1} \quad (k \in \{2, 3, \ldots, m-1\}).$$

We may assume (without loss of generality) that $\displaystyle\int_a^b \cos(f(\theta))d\theta \geq 0$. Let us first assume that $\cos(f(\theta)) \geq 0$, $(\theta \in [a, t_1])$.

Then

$$\begin{aligned} \int_a^b \cos(f(\theta))d\theta &= \int_a^{t_1} \cos(f(\theta))d\theta + \sum_{j=1}^{m-1}\int_{t_j}^{t_{j+1}} \cos(f(\theta))d\theta + \\ &\quad \int_{t_m}^b \cos(f(\theta))d\theta \\ &\leq \int_a^{t_1} \cos(f(\theta))d\theta + t_3 - t_2 + t_5 - t_4 + \ldots + \underbrace{b - t_m}_{\text{only if m is even}} \\ &\leq \frac{2}{M}\sin\left(\frac{(t_1 - a)M}{2}\right) + \frac{b - t_1}{2}. \end{aligned}$$

It follows that

$$\int_a^b \cos(f(\theta))d\theta \le \sup_{t\in[a,b]\cap[a,a+\frac{\pi}{M}]} \left(\frac{2}{M}\sin\left(\frac{(t-a)M}{2}\right)+\frac{b-t}{2}\right).$$

The supremum is attained at $t = \frac{2}{3}\pi M^{-1} + a$. The result follows.
Now, if $\cos(f(\theta)) < 0$, $(\theta \in [a, t_1])$, then

$$\int_a^b \cos(f(\theta))d\theta \le \int_{t_1}^b \cos(f(\theta))d\theta.$$

If we apply the foregoing to the R.H.S. of this last inequality we find that the result also holds in this case.

■

Lemma 2.3 *Let $m, n \in \mathbf{N}$. Let $h \in C^2[0, 2\pi]$, $K = \cup_{j=1}^m (l_j, r_j) \subseteq (0, 2\pi)$ with $r_j < l_{j+1}$ $(j \in \{1, 2, \ldots, m-1\})$ where $m \le 2n$ and $\lambda > 0$. Suppose furthermore that*

- *h'' has constant sign on (l_j, r_j) $\quad (j \in \{1, 2, \ldots, m\})$,*
- *$|h' - n| \ge \lambda n$ on (l_j, r_j) $\quad (j \in \{1, 2, \ldots, m\})$,*
- *$\mu(K) \le \frac{4}{\lambda}\arccos\left(\frac{3\sqrt{3}}{2\pi}\right)$.*

Then

$$\left|\int_K \cos(h(\theta) - n\theta)d\theta\right| \le \frac{4}{\lambda}\sin\left(\frac{\mu(K)\lambda}{4}\right).$$

Proof:
Assume that

$$\begin{aligned} |l_j - r_j| &\le \tfrac{2\pi}{3\lambda n} \quad j \in \{1, 2, \ldots, s\}, \\ |l_j - r_j| &> \tfrac{2\pi}{3\lambda n} \quad j \in \{s+1, \ldots, r\}. \end{aligned}$$

$$\begin{aligned} &\left|\int_K \cos(h(\theta) - n\theta)d\theta\right| \\ = &\left|\sum_{j=1}^m \int_{l_j}^{r_j} \cos(h(\theta) - n\theta)d\theta\right| \\ \le &\sum_{j=1}^s \frac{2}{\lambda n}\sin\left(\frac{(r_j - l_j)\lambda n}{2}\right) + \sum_{j=s+1}^m \left(\frac{\sqrt{3}}{\lambda n} + \frac{r_j - l_j - \frac{2\pi}{3\lambda n}}{2}\right) \\ \le &\frac{2s}{\lambda n}\sum_{j=1}^s \frac{1}{s}\sin\left(\frac{(r_j - l_j)\lambda n}{2}\right) + \sum_{j=s+1}^m \frac{3\sqrt{3}}{2\pi}(r_j - l_j) \\ \le &\frac{2s}{\lambda n}\sin\left(\sum_{j=1}^s \frac{(r_j - l_j)\lambda}{2s}\right) + \frac{3\sqrt{3}}{2\pi}\sum_{j=s+1}^m (r_j - l_j). \end{aligned}$$

The last inequality follows from the concavity of *sin* on $[0, \frac{\pi}{2}]$.

Let $\tau = \sum_{j=1}^{s} r_j - l_j$, then $\mu(K) = \tau + \sum_{j=s+1}^{m} r_j - l_j$.

Then

$$\begin{aligned}\left|\int_K \cos(h(\theta) - n\theta)d\theta\right| &\leq \frac{4}{\lambda}\sin\left(\frac{\tau\lambda}{4}\right) + \frac{3\sqrt{3}}{2\pi}(\mu(K) - \tau)\\ &\leq \frac{4}{\lambda}\sin\left(\frac{\mu(K)\lambda}{4}\right).\end{aligned}$$

■

Proof of Theorem 2.4.

Let $n \in \mathbf{N}$. The key to the proof is Theorem 2.2. It states that an extremal function is of the form

$$f(z) = \exp\left(-\sum_{k=1}^{n} \lambda_k \frac{1 + e^{-i\theta_k} z}{1 - e^{-i\theta_k} z}\right).$$

The real part of the n^{th} coefficient of this function is

$$\begin{aligned}\Re e\, \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f(z)}{z^{n+1}}dz &= \Re e\, \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{i\theta})}{e^{in\theta}}d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi} \Re e\left(\frac{f(e^{i\theta})}{e^{in\theta}}\right)d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi} \cos(h(\theta) - n\theta)d\theta\end{aligned}$$

where $h(\theta) = -\sum_{k=1}^{n} \lambda_k \cot\left(\frac{\theta - \theta_k}{2}\right)$.

We see that h is a piecewise differentiable, periodic function on $[0, 2\pi]$ with at most n discontinuities at points $\theta_1, \theta_2, \ldots, \theta_n$.

Moreover

$$h'(\theta) = \sum_{k=1}^{n} \frac{\lambda_k}{2} \sin^{-2}\left(\frac{\theta - \theta_k}{2}\right) \quad (\theta \notin \{\theta_1, \ldots, \theta_n\}).$$

If we want to show that $\frac{1}{2\pi}\int_0^{2\pi} \cos(h(\theta) - n\theta)d\theta$ is small, then it seems wise to show that $|h'(\theta) - n|$ is large on a substantial subset of $[0, 2\pi]$.

Therefore define

$$\begin{aligned}K_1 &= \{\theta \in [0, 2\pi] \mid h'(\theta) \geq k_1 n\},\\ K_2 &= \{\theta \in [0, 2\pi] \mid h'(\theta) \leq k_2 n\}\end{aligned}$$

where $0 < k_2 < 1 < k_1$ (see Figure 2.2).

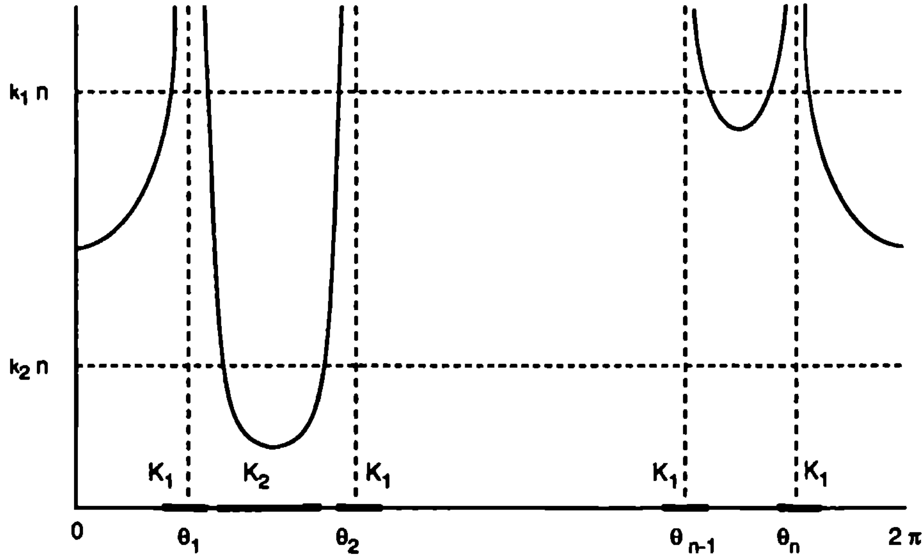

Figure 2.2

Note that, since $h'''(\theta) > 0$ on (θ_j, θ_{j+1}), $K_2 \cap (\theta_j, \theta_{j+1})$ is an interval for all j. From now on we shall include the points $\theta_1, \theta_2, \ldots, \theta_n$ in K_1. Let us first assume that

$$0 \notin K_1 \cup K_2.$$

Then both K_1 and K_2 are the union of at most n intervals:

$$\begin{aligned} K_1 &= \cup_{i \in I_1} [\alpha_i, \beta_i], \qquad I_1 \subseteq \{1, 2, \ldots, n\}, \\ K_2 &= \cup_{i \in I_2} [\gamma_i, \delta_i], \qquad I_2 \subseteq \{1, 2, \ldots, n\}. \end{aligned}$$

We first prove that

$$\frac{k_1 + k_2}{k_2} \mu(K_1) + \mu(K_2) \geq 2\pi. \tag{2.2}$$

Consider

$$\int_{[0,2\pi]-K_1} h'(\theta) d\theta = \int_{[0,2\pi]-K_1} \sum_{k=1}^{n} \frac{\lambda_k}{2} \sin^{-2}\left(\frac{\theta - \theta_k}{2}\right) d\theta.$$

Note that

$$\begin{aligned} &\int_{[0,2\pi]-K_1} \sum_{\theta_k \in (\alpha_i, \beta_i)} \frac{\lambda_k}{2} \sin^{-2}\left(\frac{\theta - \theta_k}{2}\right) d\theta \\ \leq \quad &\sum_{\theta_k \in (\alpha_i, \beta_i)} \int_{[0,2\pi]-K_1} \frac{\lambda_k}{2} \sin^{-2}\left(\frac{\theta - \theta_k}{2}\right) d\theta \end{aligned}$$

and

$$\cup_{\iota\in I_1}\{\theta_k \mid \theta_k \in (\alpha_\iota,\beta_\iota)\} = \cup_{k=1}^{n}\{\theta_k\}.$$

Therefore

$$\begin{aligned}
&\int_{[0,2\pi]-K_1} \sum_{k=1}^{n} \frac{\lambda_k}{2} \sin^{-2}\left(\frac{\theta-\theta_k}{2}\right) d\theta \\
\leq\ & \sum_{\iota\in I_1} \sum_{\theta_k\in(\alpha_\iota,\beta_\iota)} \int_{[0,2\pi]-K_1} \frac{\lambda_k}{2} \sin^{-2}\left(\frac{\theta-\theta_k}{2}\right) d\theta \\
\leq\ & \sum_{\iota\in I_1} \sum_{\theta_k\in(\alpha_\iota,\beta_\iota)} \int_{[0,2\pi]-(\alpha_\iota,\beta_\iota)} \frac{\lambda_k}{2} \sin^{-2}\left(\frac{\theta-\theta_k}{2}\right) d\theta \\
=\ & \sum_{\iota\in I_1} \sum_{\theta_k\in(\alpha_\iota,\beta_\iota)} \left|\lambda_k\left(\cot\left(\frac{\alpha_\iota-\theta_k}{2}\right) - \cot\left(\frac{\beta_\iota-\theta_k}{2}\right)\right)\right| \\
\leq\ & \sum_{\iota\in I_1} \sum_{k=1}^{n} \left|\lambda_k\left(\cot\left(\frac{\alpha_\iota-\theta_k}{2}\right) - \cot\left(\frac{\beta_\iota-\theta_k}{2}\right)\right)\right|.
\end{aligned}$$

Now note that

$$|\cot(\alpha) - \cot(\beta)| = \left|\frac{\sin(\alpha-\beta)}{\sin(\alpha)\sin(\beta)}\right| \leq \frac{|\alpha-\beta|}{2\sin^2(\alpha)} + \frac{|\alpha-\beta|}{2\sin^2(\beta)}.$$

It follows that

$$\begin{aligned}
&\int_{[0,2\pi]-K_1} h'(\theta)d\theta \\
\leq\ & \sum_{\iota\in I_1} \frac{\beta_\iota-\alpha_\iota}{2} \sum_{k=1}^{n} \frac{\lambda_k}{2}\left(\sin^{-2}\left(\frac{\alpha_\iota-\theta_k}{2}\right) + \sin^{-2}\left(\frac{\beta_\iota-\theta_k}{2}\right)\right) \\
=\ & \sum_{\iota\in I_1} \frac{\beta_\iota-\alpha_\iota}{2} \cdot 2k_1 n.
\end{aligned}$$

The last equality follows from the fact that $h'(\alpha_\iota) = h'(\beta_\iota) = k_1 n$.
Thus:

$$\int_{[0,2\pi]-K_1} h'(\theta)d\theta \leq k_1 n \mu(K_1).$$

Since

$$\mu([0,2\pi]-K_1-K_2)\cdot k_2 n \leq \int_{[0,2\pi]-K_1-K_2} h'(\theta)d\theta \leq \int_{[0,2\pi]-K_1} h'(\theta)d\theta$$

we find that

$$k_2\mu([0,2\pi]-K_1-K_2) \leq k_1\mu(K_1)$$

i.e.

$$\frac{k_1 + k_2}{k_2}\mu(K_1) + \mu(K_2) \geq 2\pi.$$

The proof of the theorem is now almost complete. Consider $h(\theta)$. On every (θ_j, θ_{j+1}) we have $h' > 0, h''' > 0$. Therefore on such intervals h'' has at most *one* zero. This implies that both K_1 and K_2 are unions of at most $2n$ intervals on which h'' is of constant sign. Let $\tilde{k}_1 = |1 - k_1|$ and $\tilde{k}_2 = |1 - k_2|$. We assume that

$$\mu(K_i) \leq 4\tilde{k}_i^{-1} \arccos\left(\frac{3\sqrt{3}}{2\pi}\right) \quad (i = 1, 2).$$

(If not, define K_i to be a subset of K_i which has this property). We now can apply Lemma 2.3 on K_1 and K_2. We find:

$$\begin{aligned}
2\pi \cdot \Re e\, a_n &\leq \left|\int_0^{2\pi} \cos(h(\theta) - n\theta)d\theta\right| \\
&\leq \left|\int_{K_1} \cos(h(\theta) - n\theta)d\theta\right| + \left|\int_{K_2} \cos(h(\theta) - n\theta)d\theta\right| + \\
&\quad \left|\int_{[0,2\pi]-(K_1 \cup K_2)} \cos(h(\theta) - n\theta)d\theta\right| \\
&\leq \frac{4}{\tilde{k}_1}\sin\left(\frac{\mu(K_1)\tilde{k}_1}{4}\right) + \frac{4}{\tilde{k}_2}\sin\left(\frac{\mu(K_2)\tilde{k}_2}{4}\right) + \\
&\quad 2\pi - \mu(K_1) - \mu(K_2).
\end{aligned}$$

We are left with an estimate with four variables; $\tilde{k}_1 \in \mathbf{R}^+$, $\tilde{k}_2 \in (0, 1)$ which we can choose freely, and $\mu(K_1)$, $\mu(K_2)$ for which we then have the constraint (2.2).
If we choose $k_1 = \frac{3}{2}$, $k_2 = \frac{1}{2}$ and note that then

$$\min(\mu(K_1), \mu(K_2)) \geq \frac{2\pi}{5}$$

we obtain the estimate as announced in the theorem.

In the foregoing we assumed that $0 \notin K_1 \cup K_2$. If we consider the proof, we see that if we identify the points 0 and 2π and note that h is periodic, the proof can easily be modified (but becomes less readable) in order to show that the theorem also holds in the case that $0 \in K_1 \cup K_2$.

∎

Note: Computer aided computations further show that the best choices for k_1, k_2 are $k_1 \approx 4.34\ldots, k_2 \approx 0.70\ldots$. It then follows that our estimate leads to a better upperbound for the constant $C(\mathbf{B}^*)$, which is smaller then 0.99509. $(1 - \frac{1}{3\pi} + \frac{4}{\pi}\sin\left(\frac{1}{12}\right) \approx 0.99988\ldots,\ \frac{4}{5} + \frac{4}{\pi}\sin\left(\frac{\pi}{20}\right) \approx 0.99918\ldots)$.

2.4 The Krzyż-conjecture for $n = 1, 2$

The Krzyż-conjecture for $n = 1, 2$ is not difficult to prove. We shall prove a more general theorem, from which the Krzyż-conjecture follows.

Consider the class $\mathbf{B}^*$. It is the subclass of the class $\mathbf{B}$ of functions omitting the value zero. Of course, the choice of the value zero is somewhat arbitrary. For $\gamma \in [0, 1)$ define:

$$\mathbf{B}^*_\gamma = \{f \in \mathbf{B} \mid f \text{ omits some } w, |w| = \gamma\}.$$

Thus: $\mathbf{B}^*_0 = \mathbf{B}^*$.
We are interested in $\mathcal{A}_n(\mathbf{B}^*_\gamma)$.

Since $\mathbf{B}^*_\gamma$ is rotation-invariant it follows that for $n \in \mathbf{N}$ we also have

$$\mathcal{A}_n(\mathbf{B}^*_\gamma) = \sup_{f \in \mathbf{B}^*_\gamma, f \text{ omits } \gamma} |a_n|.$$

Theorem 2.5

$$\mathcal{A}_n(\mathbf{B}^*_\gamma) = \frac{e^{-\alpha}(1+\alpha)^2 - e^{\alpha}(1-\alpha)^2}{2\alpha} \quad (n = 1, 2).$$

Here $\alpha \in (0, 1]$ *is such that* $\gamma = \frac{1-\alpha}{1+\alpha} e^\alpha$.

Note: The Krzyż-conjecture for $n = 1, 2$ follows, if we choose $\gamma = 0$.

Before proving the theorem, we shall deduce two lemmata needed in the proof.

Lemma 2.4 *Let* $s \in \mathbf{R}, r \in [0, 1)$. *Let* $p \in \mathbf{P}$, $p(z) = 1 + \sum_{j=1}^{\infty} c_j z^j$.

Then for all $\kappa \in \Gamma_1$*:*

$$\left| c_2 - s \frac{1 - r\kappa}{1 + r\kappa} c_1^2 \right| \le \max\left(2, \left| 4s \frac{1+r}{1-r} - 2 \right| \right).$$

Equality holds if $\kappa = -1$, $p(z) = \frac{1-z}{1+z}$.

Proof:

In Example 1.2 we proved that for all $\lambda \in \mathbf{C}$:

$$\left| c_2 + \lambda c_1^2 \right|^2 \le 4\left(1 + |c_1|^2 (|\lambda|^2 + \Re e\, \lambda)\right).$$

Now let $\lambda = -s(\frac{1-r\kappa}{1+r\kappa})$.
Then $\Re e\, \lambda = -s(\frac{1-r^2}{|1+r\kappa|^2})$. Thus:

$$\begin{aligned}
|\lambda|^2 + \Re e\, \lambda &= s^2 \left| \frac{1 - r\kappa}{1 + r\kappa} \right|^2 - s \frac{1 - r^2}{|1 + r\kappa|^2} \\
&\le \frac{s^2(1+r)^2 - s(1-r^2)}{|1 + r\kappa|^2} \\
&\le \max\left(0, \frac{s^2(1+r)^2 - s(1-r^2)}{(1-r)^2}\right)
\end{aligned}$$

i.e., in view of the inequality $|c_1| \leq 2$,

$$|c_2 + \lambda c_1^2|^2 \leq \max\left(4, 4 + 16\left(-s\frac{1+r}{1-r} + s^2\left(\frac{1+r}{1-r}\right)^2\right)\right)$$

which implies

$$|c_2 + \lambda c_1^2| \leq \max\left(2, \left|2 - 4s\frac{1+r}{1-r}\right|\right).$$

■

Lemma 2.5 *Let $k, j \in C^1[a,b]$. Suppose that*

$$j(\alpha) = -g(\alpha)k'(\alpha) = k(\alpha)h(\alpha)$$

for some $g, h \in C^1[a,b]$ and furthermore

- $k(\alpha) \geq 0,\ (\alpha \in [a,b])$,
- $g(\alpha) > 0,\ (\alpha \in [a,b])$,
- $j(\alpha)\begin{cases} < 0 & (\alpha \in [a, \alpha^*)) \\ = 0 & (\alpha = \alpha^*) \\ > 0 & (\alpha \in (\alpha^*, b]) \end{cases}$ *for some* $\alpha^* \in [a,b]$,
- $h'(\alpha)g(\alpha) \leq 1 + h(\alpha),\ (\alpha \geq \alpha^*)$.

Then

$$\sup_{\alpha \in [a,b]} j(\alpha) \leq k(\alpha^*) = \sup_{\alpha \in [a,b]} k(\alpha).$$

Proof:
Consider

$$l : \alpha \to j^3(\alpha) + k^3(\alpha),\ (\alpha \in [a,b]).$$

We claim that l is decreasing for $\alpha \geq \alpha^*$:

$$\begin{aligned} l' &= 3j^2j' + 3k^2k' \\ &= \frac{3k^3h}{g}\left(-1 + h(-h^2 + gh')\right) \\ &\leq -\frac{3k^3h}{g}(1-h)^2(1+h) \leq 0. \end{aligned}$$

It follows that for all $\alpha \in [\alpha^*, b]$:

$$j^3(\alpha) \leq j^3(\alpha^*) + k^3(\alpha^*) - k^3(\alpha) \leq k^3(\alpha^*).$$

■

Proof of Theorem 2.5.
Suppose $\gamma \in [0,1)$, $f \in \mathbf{B}^*_\gamma$, f omits γ.
Then, since $z \to \frac{z+\gamma}{1+\gamma z}$ is a bijection of $\Delta_1 \to \Delta_1$, there exists $g \in \mathbf{B}^*$ such that

$$f = \frac{g+\gamma}{1+\gamma g}.$$

Then

$$\begin{aligned} f'(0) &= (1-\gamma^2)\left(\frac{g'(0)}{(1+\gamma g(0))^2}\right), \\ f''(0) &= (1-\gamma^2)\left(\frac{g''(0)}{(1+\gamma g(0))^2} - \frac{2\gamma(g'(0))^2}{(1+\gamma g(0))^3}\right). \end{aligned}$$

Since $g \in \mathbf{B}^*$ there exists $\zeta \in \Gamma_1, \alpha > 0$ and $p \in \mathbf{P}, p(z) = \sum_{j=0}^{\infty} c_j z^j$ such that $g = \zeta \exp(-\alpha p)$. Then:

$$\begin{aligned} g(0) &= \zeta e^{-\alpha}, \\ g'(0) &= \zeta e^{-\alpha}(-\alpha c_1), \\ g''(0) &= 2\zeta e^{-\alpha}\alpha\left(\frac{\alpha}{2}c_1^2 - c_2\right). \end{aligned}$$

Therefore:

$$\begin{aligned} f(0) &= \frac{\zeta e^{-\alpha}+\gamma}{1+\gamma\zeta e^{-\alpha}}, \\ f'(0) &= -(1-\gamma^2)\frac{\zeta e^{-\alpha}\alpha c_1}{(1+\gamma\zeta e^{-\alpha})^2}, && (2.3) \\ f''(0) &= \frac{(1-\gamma^2)}{(1+\gamma\zeta e^{-\alpha})^2}\left(2\zeta e^{-\alpha}\alpha\left(\frac{\alpha}{2}c_1^2 - c_2\right) - \frac{2\gamma c_1^2(\zeta e^{-\alpha}\alpha)^2}{1+\gamma\zeta e^{-\alpha}}\right). && (2.4) \end{aligned}$$

1. $n = 1$.
   From (2.3) we see

$$|a_1| = |f'(0)| \le \sup_{\alpha \in \mathbf{R}^+} \frac{2\alpha e^{-\alpha}(1-\gamma^2)}{(1-\gamma e^{-\alpha})^2}.$$

   To find the supremum we differentiate the R.H.S. expression with respect to α. It follows that the supremum is attained if $\alpha = \alpha(\gamma)$ is such that $\gamma = \frac{1-\alpha(\gamma)}{1+\alpha(\gamma)}e^{\alpha(\gamma)}$. The stated inequality now follows. Equality is attained if

$$f(z) = \frac{\gamma - \exp\left(-\alpha(\gamma)\frac{1+z}{1-z}\right)}{1-\gamma\exp\left(-\alpha(\gamma)\frac{1+z}{1-z}\right)}.$$

2. $n = 2$.

From (2.4) we see

$$\begin{aligned} |a_2| &= \left|\frac{f''(0)}{2}\right| \\ &\leq \sup_{\alpha\in\mathbf{R}^+,\xi\in\Gamma_1,p\in\mathbf{P}} \frac{(1-\gamma^2)\alpha e^{-\alpha}}{|1+\gamma\zeta e^{-\alpha}|^2} \left|c_2 - \frac{\alpha}{2}\left(\frac{1-\gamma\xi e^{-\alpha}}{1+\gamma\xi e^{-\alpha}}\right)c_1^2\right|. \end{aligned}$$

By Lemma 2.4 we therefore find:

$$|a_2| \leq \sup_{\alpha\in\mathbf{R}^+} \frac{(1-\gamma^2)e^{-\alpha}\alpha}{(1-\gamma e^{-\alpha})^2} \max\left(2, 2\alpha\left(\frac{1+\gamma e^{-\alpha}}{1-\gamma e^{-\alpha}}\right) - 2\right). \tag{2.5}$$

For $\alpha \in \mathbf{R}^+, \gamma \in [0,1)$ define:

$$\begin{aligned} j(\gamma,\alpha) &= 2\frac{(1-\gamma^2)}{(1-\gamma e^{-\alpha})^2}e^{-\alpha}\alpha\left(\alpha\frac{1+\gamma e^{-\alpha}}{1-\gamma e^{-\alpha}} - 1\right), \\ k(\gamma,\alpha) &= 2\frac{(1-\gamma^2)}{(1-\gamma e^{-\alpha})^2}e^{-\alpha}\alpha, \\ j^*(\gamma) &= \sup_{\alpha\in\mathbf{R}^+} j(\gamma,\alpha), \\ k^*(\gamma) &= \sup_{\alpha\in\mathbf{R}^+} k(\gamma,\alpha). \end{aligned}$$

Then (2.5) can be rewritten as

$$\mathcal{A}_2(\mathbf{B}_\gamma^*) \leq \max\left(j^*(\gamma), k^*(\gamma)\right).$$

Now note that for fixed γ,

$$j : \alpha \to j(\gamma,\alpha) \text{ and } k : \alpha \to k(\gamma,\alpha)$$

satisfy the conditions of Lemma 2.5 with corresponding

$$g(\alpha) = \alpha,\ h(\alpha) = \alpha\frac{1+\gamma e^{-\alpha}}{1-\gamma e^{-\alpha}} - 1.$$

The condition: $h'(\alpha)g(\alpha) \leq 1 + h(\alpha)$ is easily checked. Therefore, for all $\gamma \in [0,1)$:

$$j^*(\gamma) \leq k^*(\gamma).$$

The proof of the theorem can now be completed by using the same argument as in the case $n = 1$. Equality is obtained if

$$f(z) = \frac{\gamma - \exp\left(-\alpha(\gamma)\frac{1+z^2}{1-z^2}\right)}{1 - \gamma\exp\left(-\alpha(\gamma)\frac{1+z^2}{1-z^2}\right)}.$$

∎

2.5 Intermezzo

In this section we solve a coefficient problem, by means of similar techniques as used in the previous section. From this, it is easy to understand how we can use these techniques to prove similar results for other coefficient problems.

Theorem 2.6 *Suppose* $f \in H(\Delta_1)$, $f(z) = \sum_{j=0}^{\infty} d_j z^j$ *and*

$$0 < r < |f(z)| < s < \infty \quad (z \in \Delta_1).$$

Then

$$|d_j| \leq 2 \frac{\exp\left(\frac{-\arctan(t)}{t}\right)}{s\sqrt{(1+t^2)}} \quad (j = 1, 2)$$

and equality is possible. Here $t = \frac{\pi}{\log(s) - \log(r)}$.

Proof:
We shall first assume that $s = 1$, and $f(0) > 0$, i.e.

$$0 < r < |f(z)| < 1, \quad (z \in \Delta_1),\ f(z) = a + d_1 z + d_2 z^2 + \ldots,\ a > 0.$$

Then $f \in \mathbf{B}^*$, and therefore there exist $\alpha > 0$, $p \in \mathbf{P}$ such that $f = \exp(-\alpha p)$. Note that

$$\alpha = -\log a,\ 0 < \Re e\, p(z) < k \quad (z \in \Delta_1)$$

where $k = \frac{\log r}{\log a}$.

Let

$$j_k : z \to \frac{-i}{\sin \frac{\pi}{k}} \left(e^{i\frac{\pi}{k}} e^{i\frac{\pi}{k}(z-1)} - \cos\frac{\pi}{k}\right).$$

Then j_k maps $\{z \in \mathbf{C} \mid 0 < \Re e\ z < k\}$ conformally onto $\{z \in \mathbf{C} \mid \Re e\ z > 0\}$ with $j_k(1) = 1$. It follows that $h = j_k \circ p \in \mathbf{P}$.

Let $\gamma = \frac{\pi}{k}$. We express the coefficients of f in those of h. Suppose $p(z) = 1 + \sum_{j=1}^{\infty} a_j z^j$, $h(z) = 1 + \sum_{j=1}^{\infty} c_j z^j$. Then

$$\begin{aligned}
f(z) &= e^{-\alpha p(z)} = e^{-\alpha}\left(1 - \alpha a_1 z + \alpha\left(-a_2 + \frac{\alpha}{2}a_1^2\right) z^2 + \ldots\right), \\
j_k(z) &= 1 + \frac{\gamma e^{i\gamma}}{\sin\gamma}(z-1) + \frac{i e^{i\gamma}\gamma^2}{\sin\gamma}\frac{(z-1)^2}{2} + \ldots, \\
h(z) &= 1 + \frac{\gamma e^{i\gamma}}{\sin\gamma} a_1 z + \frac{\gamma e^{i\gamma}}{\sin\gamma}\left(a_2 + \frac{i\gamma a_1^2}{2}\right) z^2 + \ldots.
\end{aligned}$$

It follows that

$$\begin{aligned} |d_1| &= \alpha e^{-\alpha}\frac{\sin\gamma}{\gamma}|c_1|, \\ |d_2| &= \alpha e^{-\alpha}\frac{\sin\gamma}{\gamma}\left|c_2 - \frac{\beta}{2}c_1^2\right| \end{aligned}$$

where $\beta = e^{-i\gamma}(i\sin\gamma + \frac{\alpha\sin\gamma}{\gamma})$.
Note that $\gamma = \frac{\pi\log a}{\log r} = -\frac{\pi\alpha}{\log r}$. Let $t = -\frac{\pi}{\log r}$.

1. $n = 1$.
   $|d_1| = \frac{2}{t}e^{-\alpha}\sin t\alpha$.
   If we maximize the R.H.S. expression over $\alpha \in (0, \frac{\pi}{t})$ we find the (sharp) estimate
   $$|d_1| \le 2\frac{\exp\left(-\frac{\arctan t}{t}\right)}{\sqrt{(1+t^2)}}.$$

2. $n = 2$.
   $|d_2| = \frac{1}{t}e^{-\alpha}|c_2 - \frac{\beta}{2}c_1^2|\sin t\alpha$.
   From Example 1.2 it follows that (note that $|c_1| \le 2$)
   $$|c_2 - \frac{\beta}{2}c_1^2| \le 2\max(1, |\beta - 1|).$$
   $|\beta - 1| = |\frac{\alpha\sin\gamma}{\gamma} - \cos\gamma|$. Therefore:
   $$|d_2| = \frac{2}{t}e^{-\alpha}\sin t\alpha\max(1, \frac{\alpha\sin\gamma}{\gamma} - \cos\gamma). \tag{2.6}$$
   For $t \in [0, \infty)$, $\alpha \in (0, \frac{\pi}{t})$ define
   $$\begin{aligned} j(t,\alpha) &= \frac{2}{t}e^{-\alpha}\sin t\alpha\left(\frac{\sin t\alpha}{t} - \cos t\alpha\right), \\ k(t,\alpha) &= \frac{2}{t}e^{-\alpha}\sin t\alpha, \\ j^*(t) &= \sup_{\alpha\in[0,\frac{\pi}{t}]} j(t,\alpha), \\ k^*(t) &= \sup_{\alpha\in[0,\frac{\pi}{t}]} k(t,\alpha). \end{aligned}$$
   Then (2.6) can be rewritten as
   $$|d_2| = \max\left(j^*(t), k^*(t)\right).$$
   Now note that for fixed t,
   $$j : \alpha \to j(t,\alpha) \text{ and } k : \alpha \to k(t,\alpha)$$

satisfy the conditions of Lemma 2.5 with corresponding

$$g(\alpha) = \frac{\sin t\alpha}{t},\ h(\alpha) = \frac{\sin t\alpha}{t} - \cos t\alpha.$$

The condition $h'(\alpha)g(\alpha) \leq 1 + h(\alpha)$ is also satisfied since:

$$\begin{aligned} h'(\alpha)g(\alpha) - h(\alpha) &= \frac{\sin t\alpha}{t}(\cos t\alpha + t\sin t\alpha) - \frac{\sin t\alpha}{t} + \cos t\alpha \\ &= 1 - \cos^2 t\alpha + \frac{1}{t}\sin t\alpha\cos t\alpha - \frac{\sin t\alpha}{t} + \cos t\alpha \\ &= 1 - h(\alpha)(1 - \cos t\alpha) \leq 1 \end{aligned}$$

Therefore for all $t \in [0, \infty)$:

$$|d_2| \leq 2\frac{\exp\left(-\frac{\arctan t}{t}\right)}{\sqrt{(1+t^2)}}.$$

The proof of the theorem is now easily completed:

Suppose

$$0 < r < |f(z)| < s < \infty, \quad (z \in \Delta_1),\ f(0) = \xi.$$

Then $\tilde{f} : z \to \frac{\bar{\xi}f(z)}{|\xi|s}$ is a function as considered above.
(With corresponding $t = \frac{\pi}{-\log(\frac{r}{s})} = \frac{\pi}{\log s - \log r}$).
From this the conjectured estimates follow.
Equality is attained if

$$\begin{aligned} f(z) &= e^{\frac{\log r}{\pi i}\log(i\frac{z-1}{z+1}\sin\pi k + \cos\pi k)} && \text{(maximize } d_1\text{)}, \\ f(z) &= e^{\frac{\log r}{\pi i}\log(i\frac{z^2-1}{z^2+1}\sin\pi k + \cos\pi k)} && \text{(maximize } d_2\text{)} \end{aligned}$$

where $\alpha = \frac{\log s - \log r}{\pi}\arctan\frac{\pi}{\log s - \log r}$ and $k = -\frac{\log r}{\alpha}$.
Both functions map Δ_1 *onto* $\{z \in \mathbf{C} \mid r < |z| < s\}$.

∎

2.6 The Krzyż-conjecture for $n = 3$

We now return to the class $\mathbf{B}^*$.
In order to prove the Krzyż-conjecture for $n = 3$ we shall express the coefficients of functions in $\mathbf{B}^*$ in those of functions in class $\mathbf{B}$ and $\mathbf{P}$.

Since $\mathbf{B}^*$ is rotation-invariant we assume

$$f(z) = \sum_{j=0}^{\infty} a_j z^j,\ a_0 \geq 0, a_3 \geq 0.$$

1. Let $p \in \mathbf{P}$, $p(z) = 1 + \sum_{j=1}^{\infty} c_j z^j$ be such that $f = \exp(-\alpha p)$, $\alpha = -\log(f(0))$. Then

$$a_3 = \alpha e^{-\alpha}\left(-\frac{\alpha^2}{6}c_1^3 + \alpha c_1 c_2 - c_3\right). \tag{2.7}$$

2. Let $w \in \mathbf{B}_0, w(z) = \sum_{j=1}^{\infty} d_j z^j$ be such that $f = \exp\left(\alpha \frac{w-1}{w+1}\right)$, $\alpha = -\log(f(0))$. Then

$$a_3 = 2\alpha e^{-\alpha}\left(d_3 + (2\alpha - 2)d_1 d_2 + \left(\frac{2}{3}\alpha^2 - 2\alpha + 1\right)d_1^3\right). \tag{2.8}$$

Theorem 2.7 *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j \in \mathbf{B}^*, a_3 \geq 0, a_0 = e^{-\alpha}, \alpha \geq 0$. *Then*

$$a_3 \leq \begin{cases} 2\alpha e^{-\alpha} & \alpha \leq 1\frac{1}{2} \\ 2\alpha e^{-\alpha}\sqrt{3\left(\frac{-\alpha^2+2\alpha-1}{\alpha^2-6\alpha+6}\right)} & \alpha \in [1\frac{1}{2}, S] \\ 2\alpha e^{-\alpha}\left(\frac{2}{3}\alpha^2 - 2\alpha + 1\right) & \alpha \geq S. \end{cases}$$

Here $S \approx 4.06\ldots$ *is the only real root of the polynomial* $2\alpha^3 - 12\alpha^2 + 18\alpha - 9$. *These inequalities are sharp for* $\alpha \notin (1\frac{1}{2}, S)$.

Note: We can obtain a sharp inequality for $\alpha \in (1\frac{1}{2}, S)$ by using a result due to Campschroer [4], p. 34 (See also Prokhorov and Szynal [24]). However, the results from the theorem are strong enough to prove the following

Corollary 2.1 *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j \in \mathbf{B}^*$. *Then*

$$|a_3| \leq \frac{2}{e}.$$

We have equality if and only if f is of the form

$$f(z) = \kappa \exp\left(\frac{\xi z^3 - 1}{\xi z^3 + 1}\right), \quad \kappa, \xi \in \Gamma_1.$$

Proof of Theorem 2.7.

1. From (2.7) we have $a_3 = \alpha e^{-\alpha}\left|-\frac{\alpha^2}{6}c_1^3 + \alpha c_1 c_2 - c_3\right|$. We estimate the R.H.S. with aid of Theorem 1.15. We have:

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ & c_1 & c_2 \\ \emptyset & & c_1 \end{pmatrix} \ll \begin{pmatrix} 2 & c_1 & c_2 \\ & 2 & c_1 \\ \emptyset & & 2 \end{pmatrix}.$$

Let $\vec{x} = \left(\frac{\alpha^2}{6}c_1^2 - c_2, (1-\alpha)c_1, 1\right) \in \mathbf{C}^3$. We find:

$$\begin{aligned} & \left|-\frac{\alpha^2}{6}c_1^3 + \alpha c_1 c_2 \quad c_3\right|^2 + \left|(1-\alpha)c_1^2 - c_2\right|^2 + |c_1|^2 \\ \leq & \left|\left(\frac{\alpha^2}{3} - \alpha + 1\right) c_1^2 - c_2\right|^2 + |(3-2\alpha)c_1|^2 + 4. \end{aligned}$$

From this it follows that

$$\begin{aligned} \left|-\tfrac{\alpha^2}{6}c_1^3 + \alpha c_1 c_2 - c_3\right|^2 \quad \leq \quad & 4 + \left((3-2\alpha)^2 - 1\right)|c_1|^2 + \\ & \left(\left(\tfrac{\alpha^2}{3} - \alpha + 1\right)^2 - (\alpha-1)^2\right)|c_1|^4 + \\ & 2\,|c_2|\,|c_1|^2 \left|\tfrac{\alpha^2}{3} - 2\alpha + 2\right|. \end{aligned}$$

Since $|c_2| \leq 2$ we therefore find:

Estimate 2.1

$$a_3^2 < \alpha^2 e^{-2\alpha}(4 + Ax + Bx^2)$$

where

$$\begin{aligned} x &= |c_1|^2 \in [0,4] \\ A &= (3-2\alpha)^2 - 1 + 4\left|\frac{\alpha^2}{3} - 2\alpha + 2\right| \\ B &= \frac{\alpha^4}{9} - \frac{2\alpha^3}{3} + \frac{2\alpha^2}{3}. \end{aligned}$$

2. From (2.8) we have

$$a_3 = 2\alpha e^{-\alpha}\left|d_3 + (2\alpha - 2)d_1 d_2 + \left(\frac{2}{3}\alpha^2 - 2\alpha + 1\right) d_1^3\right|.$$

We estimate the R.H.S. with aid of Theorem 1.14:
We have

$$\begin{pmatrix} d_1 & d_2 & d_3 \\ & d_1 & d_2 \\ \emptyset & & d_1 \end{pmatrix} \ll \begin{pmatrix} 1 & & \emptyset \\ & 1 & \\ \emptyset & & 1 \end{pmatrix}.$$

Let $\vec{x}_{\rho\lambda} = \left(\rho(d_2 + \lambda d_1^2), \lambda d_1, 1\right) \in \mathbf{C}^3$, $\quad (\rho, \lambda \subset \mathbf{C})$. We find:

$$\left|d_3 + (\rho + \lambda)d_1 d_2 + \rho\lambda d_1^3\right|^2 \leq 1 + (|\lambda|^2 - 1)|d_1|^2 + (|\rho|^2 - 1)\left|d_2 + \lambda d_1^2\right|^2.$$

Now apply Pick's Lemma ($|d_2| \leq 1 - |d_1|^2$):

Estimate 2.2

$$\left|d_3 + (\rho+\lambda)d_1d_2 + \rho\lambda d_1^3\right|^2 \le \begin{cases} |\rho\lambda|^2 & \text{if } |\rho| \ge 1, |\lambda| \ge 1, \\ 1 & \text{if } |\rho| \le 1, |\lambda| \le 1. \end{cases}$$

(a) Assume that $\alpha \le 3 - \sqrt{3}$. We use Estimate 2.2.

$$\begin{aligned} \text{Choose } \rho &= \alpha(1 + \tfrac{1}{3}\sqrt{3}) - 1 \\ \lambda &= \alpha(1 - \tfrac{1}{3}\sqrt{3}) - 1. \end{aligned}$$

Then $|\rho| \le 1, |\lambda| \le 1$. Therefore

$$a_3 = 2\alpha e^{-\alpha}\left|d_3 + (\rho+\lambda)d_1d_2 + \rho\lambda d_1^3\right| \le 2\alpha e^{-\alpha}.$$

(b) Assume that $\alpha \in (3-\sqrt{3}, 3+\sqrt{3})$. We use Estimate 2.1.
Note that $B = \frac{\alpha^2}{3}(\frac{\alpha^2}{3} - 2\alpha + 2) \le 0$. Therefore

$$x \to 4 + Ax + Bx^2$$

attains its maximum for $x = -\frac{1}{2}AB^{-1}$. However, since $x \in [0,4]$ it now follows that

(b1) $\alpha \in (3-\sqrt{3}, 1\frac{1}{2}]$:
$x = 0$ maximizes the R.H.S. of Estimate 2.1.

(b2) $\alpha \in [1\frac{1}{2}, S]$:
$x = -\frac{1}{2}AB^{-1}$ maximizes the R.H.S. of Estimate 2.1.

(b3) $\alpha \in [S, 3+\sqrt{3})$:
$x = 4$ maximizes the R.H.S. of Estimate 2.1.

Here S is the positive zero of $-A = 8B$, i.e. $2S^3 - 12S^2 + 18S - 9 = 0$.

(c) Assume that $\alpha \ge 3 + \sqrt{3}$. We again use Estimate 2.2. The same choices for ρ and λ as in (a) now yield:

$$|\rho| \ge 1, |\lambda| \ge 1$$

and therefore:

$$a_3 \le 2\alpha e^{-\alpha}\left(\frac{2}{3}\alpha^2 - 2\alpha + 1\right).$$

The estimates as announced in the theorem now follow.

Sharpness is attained for

$$\begin{aligned} f(z) &= \exp\left(\alpha \tfrac{z^3-1}{z^3+1}\right) && (\alpha \le 1\tfrac{1}{2}) \\ f(z) &= \exp\left(\alpha \tfrac{z-1}{z+1}\right) && (\alpha \ge S). \end{aligned}$$

■

Proof of Corollary 2.1
Consider the estimates of Theorem 2.7.

(a) Suppose $\alpha \le 1\frac{1}{2}$.
Then $a_3 \le 2\alpha e^{-\alpha} \le \frac{2}{e}$ with equality only if $\alpha = 1$.

(b) Suppose $\alpha \in [1\frac{1}{2}, S]$.
Then $a_3 \le 2\alpha e^{-\alpha} \frac{(\alpha-1)}{\sqrt{h(\alpha)}}$ where $h(\alpha) = -\frac{\alpha^2}{3} + 2\alpha - 2$.
Note that, since

$$\frac{d}{d\alpha}\left(2\alpha e^{-\alpha}\frac{(\alpha-1)}{\sqrt{h(\alpha)}}\right) = h^{-\frac{3}{2}}(\alpha)\frac{e^{-\alpha}}{3}\left(2(\alpha-2)(\alpha^3-6\alpha^2+9\alpha-3)\right)$$

we only need to consider the cases $\alpha = 1\frac{1}{2}, \alpha = 2, \alpha = S$. Since the cases $\alpha = 1\frac{1}{2}, \alpha = S$ correspond with estimates considered in (a) and (c) it satisfies to verify $\alpha = 2$:

$$\left(2\alpha e^{-\alpha}\frac{(\alpha-1)}{\sqrt{h(\alpha)}}\right)_{|\alpha=2} = 2\sqrt{6}e^{-2} < \frac{2}{e}.$$

(c) Suppose $\alpha \ge S$.
Then

$$a_3 \le 2\alpha e^{-\alpha}\left(\frac{2}{3}\alpha^2 - 2\alpha + 1\right) \le 2\alpha e^{-\alpha}\left(\frac{2}{3}\alpha^2 - 2\alpha + \frac{16}{9}\right).$$

Since the R.H.S. of the previous inequality is decreasing for $\alpha \ge 4$ it follows that

$$2\alpha e^{-\alpha}\left(\frac{2}{3}\alpha^2 - 2\alpha + 1\right) \le \frac{160}{9}e^{-4} < \frac{2}{e}.$$

From this we see that, if $a_3 = \frac{2}{e}$, then $\alpha = 1$. From the proof of Theorem 2.7 we see that both d_1 and $d_2 + \lambda d_1^2$ must be zero, and $|d_3| = 1$. From the maximum modulus principle it then follows that $w(z) = \xi z^3$, $|\xi| = 1$. But this implies that

$$f(z) = \kappa \exp\left(\frac{\xi z^3 - 1}{\xi z^3 + 1}\right), \quad \kappa, \xi \in \Gamma_1.$$

■

2.7 The Krzyż-conjecture for $n = 4$

In order to prove the Krzyż-conjecture for $n = 4$, we shall express the coefficients of a function $f \in \mathbf{B}^*$ in those of functions $p \in \mathbf{P}$. From this we shall deduce some estimates for the fourth coefficient. In section 2.7.2 we then show that from these estimates the conjecture follows.

2.7.1 Estimates

If $f \in \mathbf{B}^*, f(0) > 0$, there exist $p \in \mathbf{P}$ and $\alpha \geq 0$ such that $f = \exp(-\alpha p)$. Furthermore, by Theorem 1.5, f is subordinate to $k_\alpha : z \to \exp\left(-\alpha \frac{1+z}{1-z}\right)$. Write $f(z) = \sum_{j=0}^{\infty} a_j z^j$, $p(z) = 1 + \sum_{j=1}^{\infty} c_j z^j$, $k_\alpha(z) = \sum_{j=0}^{\infty} b_j z^j$. It follows from Rogosinski's Theorem 1.3 that

$$|a_4|^2 \leq \sum_{k=1}^{4} |a_k|^2 \leq \sum_{k=1}^{4} |b_k|^2.$$

We find:

Estimate 2.3

$$|a_4| \leq \frac{2}{3} e^{-\alpha} \alpha \sqrt{\alpha^6 - 12\alpha^5 + 58\alpha^4 - 138\alpha^3 + 174\alpha^2 - 108\alpha + 36}$$

If we express the coefficients of f in those of p we find

$$|a_4| = \alpha e^{-\alpha} \left| \frac{\alpha^3}{24} c_1^4 - \frac{\alpha^2}{2} c_1^2 c_2 + \alpha c_1 c_3 + \frac{\alpha}{2} c_2^2 - c_4 \right|. \tag{2.9}$$

We shall estimate the R.H.S. of (2.9) in terms of $x = |c_1|^2$, $y = |c_2|$ and α.

By Theorem 1.15 we have

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ & c_1 & c_2 & c_3 \\ & & c_1 & c_2 \\ \emptyset & & & c_1 \end{pmatrix} \ll \begin{pmatrix} 2 & c_1 & c_2 & c_3 \\ & 2 & c_1 & c_2 \\ & & 2 & c_1 \\ \emptyset & & & 2 \end{pmatrix}. \tag{2.10}$$

Let $q \in \mathbf{R}$, $A = -\frac{\alpha^2}{2} - \frac{q}{4}$. Let

$$\vec{u} = \left(\frac{\alpha^3}{24} c_1^3 + A c_1 c_2 + \frac{\alpha}{2} c_3, \frac{q}{4} c_1^2 + \frac{\alpha}{2} c_2, \frac{\alpha}{2} c_1, -1 \right) \in \mathbf{C}^4.$$

We find:

$$\begin{aligned}
&\left|\frac{\alpha^3}{24}c_1^4 - \frac{\alpha^2}{2}c_1^2c_2 + \alpha c_1c_3 + \frac{\alpha}{2}c_2^2 - c_4\right|^2 + \\
&\left|\frac{q}{4}c_1^3 + \alpha c_1c_2 - c_3\right|^2 + \left|\frac{\alpha}{2}c_1^2 - c_2\right|^2 + |c_1|^2 \\
\leq\ &\left|\left(\frac{\alpha^3}{12} + \frac{q}{4}\right)c_1^3 + (2A+\alpha)c_1c_2 + (\alpha-1)c_3\right|^2 + \\
&\left|\left(\frac{q}{2} + \frac{\alpha}{2}\right)c_1^2 + (\alpha-1)c_2\right|^2 + |(\alpha-1)c_1|^2 + 4.
\end{aligned} \tag{2.11}$$

By Theorem 1.15 we also have:

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ & c_1 & c_2 \\ \emptyset & & c_1 \end{pmatrix} \ll \begin{pmatrix} 2 & c_1 & c_2 \\ & 2 & c_1 \\ \emptyset & & 2 \end{pmatrix}.$$

Let

$$\vec{v} = \left(\left(\frac{\alpha^3}{12} + \frac{q}{4}\right)c_1^2 + \left(-\alpha^2 - q + \frac{\alpha}{2}\right)c_2, \left(\frac{q}{2} + \frac{\alpha}{2}\right)c_1, \alpha - 1\right) \in \mathbf{C}^3.$$

Note that $2A + \alpha = (-\alpha^2 - q + \frac{\alpha}{2}) + (\frac{q}{2} + \frac{\alpha}{2})$. We find

$$\begin{aligned}
&\left|\frac{\alpha^3}{24}c_1^4 - \frac{\alpha^2}{2}c_1^2c_2 + \alpha c_1c_3 + \frac{\alpha}{2}c_2^2 - c_4\right|^2 \\
\leq\ &4 + (2\alpha-2)^2 + \left|\left(\frac{\alpha^3}{6} + q + \frac{\alpha}{2}\right)c_1^2 + (-2q + 2\alpha - 1 - 2\alpha^2)c_2\right|^2 + \\
&\left((q+2\alpha-1)^2 - 1\right)|c_1|^2 - \left|\frac{\alpha}{2}c_1^2 - c_2\right|^2 - \left|\frac{q}{4}c_1^3 - \alpha c_1c_2 - c_3\right|^2.
\end{aligned} \tag{2.12}$$

From these inequalities we shall derive three estimates:

1. Choose $q = \alpha - \alpha^2 - \frac{4}{5}$. We find

Estimate 2.4

$$\left|\frac{\alpha^3}{24}c_1^4 - \frac{\alpha^2}{2}c_1^2c_2 + \alpha c_1c_3 + \frac{\alpha}{2}c_2^2 - c_4\right|^2 \leq h_1(\alpha, x, y)$$

where

$$\begin{aligned}
h_1(\alpha, x, y) &= \mathbf{A}_1 + \mathbf{B}_1x^2 + \mathbf{C}_1y^2 + \mathbf{D}_1x + \mathbf{E}_1xy \\
x &= |c_1|^2 \in [0, 4]
\end{aligned}$$

$$
\begin{aligned}
y &= |c_2| \in [0,2] \\
\mathbf{A}_1 &= 4 + (2\alpha - 2)^2 \\
\mathbf{B}_1 &= \mu^2 - \frac{\alpha^2}{4} \\
\mathbf{C}_1 &= -\frac{16}{25} \\
\mathbf{D}_1 &= \left(3\alpha - \alpha^2 - 1\frac{4}{5}\right)^2 - 1 \\
\mathbf{E}_1 &= |1\frac{1}{5}\mu + \alpha| \\
\mu &= \frac{\alpha^3}{6} - \alpha^2 + 1\frac{1}{2}\alpha - \frac{4}{5}.
\end{aligned}
$$

2. If we apply Example 1.2 to (2.12) we find

$$
\begin{aligned}
&\left|\frac{\alpha^3}{24}c_1^4 - \frac{\alpha^2}{2}c_1^2c_2 + \alpha c_1c_3 + \frac{\alpha}{2}c_2^2 - c_4\right|^2 \\
\le\ & 4 + (2\alpha - 2)^2 + 4D^2 + x(4C^2 + 4CD + E^2 - 1)
\end{aligned}
$$

where

$$
\begin{aligned}
x &= |c_1|^2 \in [0,4] \\
C &= q + \frac{\alpha^3}{6} + \frac{\alpha}{2} \\
D &= -2q + 2\alpha - 1 - \alpha^2 \\
E &= q + 2\alpha - 1.
\end{aligned}
$$

Rewrite this to

$$
\left|\frac{\alpha^3}{24}c_1^4 - \frac{\alpha^2}{2}c_1^2c_2 + \alpha c_1c_3 + \frac{\alpha}{2}c_2^2 - c_4\right|^2 \le g(\alpha, x)
$$

where

$$
\begin{aligned}
g(\alpha, x) &= (\mathbf{R}_1 + \mathbf{S}_1 q + \mathbf{T}_1 q^2) + x(\mathbf{R}_2 + \mathbf{S}_2 q + \mathbf{T}_2 q^2) \\
x &= |c_1|^2 \in [0,4] \\
\mathbf{R}_1 &= 4 + (2\alpha - 2)^2 + 4(2\alpha - 1 - 2\alpha^2)^2 \\
\mathbf{S}_1 &= -16(2\alpha - 1 - 2\alpha^2) \\
\mathbf{T}_1 &= 16 \\
\mathbf{R}_2 &= 4\left(\left(\frac{\alpha^3}{6} + \frac{\alpha}{2}\right)\left(\frac{\alpha^3}{6} - 2\alpha^2 + 2\frac{1}{2}\alpha - 1\right) + \alpha^2 - \alpha\right) \\
\mathbf{S}_2 &= 12\alpha - 6 - 8\alpha^2 \\
\mathbf{T}_2 &= -3.
\end{aligned}
$$

For fixed α and x, the sharpest estimate for g is obtained if we choose

$$q = -\frac{\mathbf{S}_1 + \mathbf{S}_2 x}{2(\mathbf{T}_1 + \mathbf{T}_2 x)}.$$

Therefore

$$\left| \frac{\alpha^3}{24} c_1^4 - \frac{\alpha^2}{2} c_1^2 c_2 + \alpha c_1 c_3 + \frac{\alpha}{2} c_2^2 - c_4 \right|^2 \le \mathbf{R}_1 + \mathbf{R}_2 x - \frac{(\mathbf{S}_1 + \mathbf{S}_2 x)^2}{4(\mathbf{T}_1 + \mathbf{T}_2 x)}$$

i.e.

Estimate 2.5

$$\left| \frac{\alpha^3}{24} c_1^4 - \frac{\alpha^2}{2} c_1^2 c_2 + \alpha c_1 c_3 + \frac{\alpha}{2} c_2^2 - c_4 \right|^2 \le h_2(\alpha, x)$$

where

$$\begin{aligned}
h_2(\alpha, x) &= \mathbf{A}_2 + \mathbf{B}_2 x + \frac{\mathbf{C}_2}{(16 - 3x)} \\
x &= |c_1|^2 \in [0, 4] \\
\mathbf{A}_2 &= 4 + (2\alpha - 2)^2 + 4\left(\frac{2}{3}\alpha^2 - 2\alpha + 1\right)^2 \\
\mathbf{B}_2 &= \frac{1}{9}(\alpha^6 - 12\alpha^5 + 66\alpha^4 - 186\alpha^3 + 261\alpha^2 - 162\alpha + 27) \\
\mathbf{C}_2 &= -64\left(\frac{2}{3}\alpha^2 - 2\alpha + 1\right)^2.
\end{aligned}$$

3. Apply (2.10) to

$$\vec{w} = \left(\frac{\alpha^3}{24} c_1^3 + \left(\frac{\alpha^3}{12} - \frac{\alpha^2}{2}\right) c_1 c_2 + \frac{1}{2} c_3, -\frac{\alpha^3}{12} c_1^2 + \frac{\alpha}{2} c_2, \left(\alpha - \frac{1}{2}\right) c_1, -1 \right).$$

Then

$$\begin{aligned}
&\left| \frac{\alpha^3}{24} c_1^4 - \frac{\alpha^2}{2} c_1^2 c_2 + \alpha c_1 c_3 + \frac{\alpha}{2} c_2^2 - c_4 \right|^2 + \left| \left(\alpha - \frac{1}{2}\right) c_1^2 - c_2 \right|^2 + \\
&\left| -\frac{\alpha^3}{12} c_1^3 + \left(1\frac{1}{2}\alpha - \frac{1}{2}\right) c_1 c_2 - c_3 \right|^2 + |c_1|^2 \\
\le\ &\left| \left(\frac{\alpha^3}{6} - \alpha^2 + 1\frac{1}{2}\alpha - \frac{1}{2}\right) c_1 c_2 \right|^2 + |(2\alpha - 2) c_1|^2 + \\
&\left| \left(-\frac{\alpha^3}{6} + 2\alpha - 1\right) c_1^2 + (\alpha - 1) c_2 \right|^2 + 4.
\end{aligned}$$

From this we find:

Estimate 2.6

$$\left|\frac{\alpha^3}{24}c_1^4 - \frac{\alpha^2}{2}c_1^2c_2 + \alpha c_1c_3 + \frac{\alpha}{2}c_2^2 - c_4\right|^2 \leq h_3(\alpha, x, y)$$

where

$$\begin{aligned}
h_3(\alpha, x, y) &= \mathbf{A}_3 + \mathbf{B}_3x^2 + \mathbf{C}_3y^2 + \mathbf{D}_3x + \mathbf{E}_3xy + \mathbf{F}_3xy^2 \\
x &= |c_1|^2 \in [0, 4] \\
y &= |c_2| \in [0, 2] \\
\mathbf{A}_3 &= 4 \\
\mathbf{B}_3 &= \frac{\alpha^6}{36} - \frac{\alpha^4}{3} + \frac{\alpha^3}{6} \\
\mathbf{C}_3 &= \alpha^2 - 2\alpha \\
\mathbf{D}_3 &= 4\alpha^2 - 8\alpha + 3 \\
\mathbf{E}_3 &= 2\left|-\frac{\alpha^4}{6} + \frac{\alpha^3}{6} + \alpha^2 - \frac{\alpha}{2}\right| \\
\mathbf{F}_3 &= \left(-\alpha^2 + \frac{\alpha^3}{6} + 1\frac{1}{2}\alpha - \frac{1}{2}\right)^2.
\end{aligned}$$

2.7.2 Estimation

Equiped with the estimates of the previous section we can now tackle the Krzyż-conjecture for $n = 4$. In the following we shall state some inequalities concerning polynomials of low degree without giving a proof. These inequalities will be marked †. (Although the stated inequalities are not trivial, one easily shows that they hold, using routine calculus).

From (2.9) we see that it suffices to show that

$$\left|\frac{\alpha^3}{24}c_1^4 - \frac{\alpha^2}{2}c_1^2c_2 + \alpha c_1c_3 + \frac{\alpha}{2}c_2^2 - c_4\right|^2 \leq \frac{4}{\alpha^2}\exp(2\alpha - 2).$$

Let

$$\nu(\alpha) = \frac{4}{\alpha^2}\exp(2\alpha - 2).$$

Note that the equation

$$4 + (2\alpha - 2)^2 = \nu(\alpha)$$

has two roots: $\alpha = 1$ and $\alpha^* \approx 2.54\ldots$. To simplify our notations we further introduce

$$\begin{aligned}
\tilde{\mathbf{A}}_1 &= \mathbf{A}_1 - \nu(\alpha), \\
\tilde{\mathbf{A}}_2 &= \mathbf{A}_2 - \nu(\alpha), \\
\tilde{\mathbf{A}}_3 &= \mathbf{A}_3 - \nu(\alpha).
\end{aligned}$$

1. It is easy to verify that if $\alpha < 0.66$ or $\alpha > 3.61$ then the R.H.S. of Estimate 2.3 is less than $\frac{2}{e}$. Therefore we may assume $\alpha \in [0.66, 3.61]$.
Consider Estimate 2.5. We have $\frac{\partial}{\partial x}h_2(\alpha, x) = \mathbf{B}_2 + \frac{3\mathbf{C}_2}{(16-3x)^2}$.

   (a) Suppose $\mathbf{B}_2 + \frac{3\mathbf{C}_2}{256} \leq 0$. This is the case if $\alpha \in [0.22, 2.61]^\dagger$.
   Then:
   $$h_2(\alpha, x) \leq h_2(\alpha, 0) = 4 + (2\alpha - 2)^2.$$

   (b) Suppose $\mathbf{B}_2 + \frac{3\mathbf{C}_2}{256} > 0$. Note that
   $$h_2(\alpha, x) \leq \nu(\alpha) \Leftrightarrow f(x) \leq 0$$
   where
   $$f(x) = 16\tilde{\mathbf{A}}_2 + \mathbf{C}_2 + x(16\mathbf{B}_2 - 3\tilde{\mathbf{A}}_2) - 3x^2\mathbf{B}_2.$$
   Since
   $$\begin{aligned} f(x) &\leq 16\tilde{\mathbf{A}}_2 + \mathbf{C}_2 + (16\mathbf{B}_2 - 3\tilde{\mathbf{A}}_2)^2(12\mathbf{B}_2)^{-1} \\ &= \mathbf{C}_2 + (16\mathbf{B}_2 + 3\tilde{\mathbf{A}}_2)^2(12\mathbf{B}_2)^{-1} \\ &< 0 \end{aligned}$$
   throughout the interval $[2.60, 7.72]^\dagger$ we conclude that:

Theorem 2.8 *If* $\alpha \notin (1, \alpha^*]$ *then*
$$|a_4| \leq \frac{2}{e}$$
with equality only if $\alpha = 1, a_1 = 0$.

■

2. Now suppose $\alpha \in (1, \alpha^*]$. We shall prove the following theorem:

Theorem 2.9
$$\forall_{\alpha\in(1,\alpha^*]}\forall_{x\in[0,4]}\forall_{y\in[0,2]} : \min(h_1(\alpha, x, y), h_3(\alpha, x, y)) < \nu(\alpha).$$

Proof:
Define
$$\begin{aligned} g_1(\alpha, x, y) &= h_1(\alpha, x, y) - \nu(\alpha) \\ g_2(\alpha, x, y) &= h_3(\alpha, x, y) + \mathbf{F}_3(1 - x)y^2 - \nu(\alpha) \\ G_1 &= \{(x, y) | x \in [\frac{3}{4}, 4],\ y \in [0, 2]\} \\ G_2 &= \{(x, y) | x \in [0, \frac{3}{4}],\ y \in [0, 2]\} \\ G_3 &= \{(x, y) | 2x + y \geq r,\ (x, y) \in G_2\} \\ G_4 &= \{(x, y) | 2x + y \leq r,\ x \geq 0,\ y \geq 0\} \\ r &= 2\frac{1}{3}. \end{aligned}$$

We prove the following:

(a) If $\alpha \in (1, \alpha^*]$ and $(x, y) \in G_1$, then $g_1(\alpha, x, y) < 0$.
(b) If $\alpha \in (1, 1\frac{1}{2}]$ and $(x, y) \in G_2$, then $g_2(\alpha, x, y) < 0$.
(c) If $\alpha \in [1\frac{1}{2}, \alpha^*]$ and $(x, y) \in G_3$, then $g_1(\alpha, x, y) < 0$.
(d) If $\alpha \in [1\frac{1}{2}, \alpha^*]$ and $(x, y) \in G_4$, then $g_2(\alpha, x, y) < 0$.

From this the theorem follows.

Remark: In the following we shall use the following estimates:†

(E.1)	$16\mathbf{B}_1\mathbf{C}_1 - 4\mathbf{E}_1^2 + 2\mathbf{D}_1\mathbf{C}_1 > 0$	$\alpha \in (1, \alpha^*]$
(E.2)	$(3\frac{3}{4}\mu + 1\frac{1}{8}\alpha)^2 + 12\mathbf{D}_1 + 16\tilde{\mathbf{A}}_1 < 0$	$\alpha \in (1, \alpha^*]$
(E.3)	$4(\mathbf{C}_3 + \mathbf{F}_3)\mathbf{B}_3 - \mathbf{E}_3^2 < 0$	$\alpha \in (1, 1\frac{1}{2}]$
(E.4)	$\frac{9}{16}\left(\mathbf{B}_3 - \frac{\mathbf{E}_3^2}{4(\mathbf{C}_3+\mathbf{F}_3)}\right) + \frac{3}{4}\mathbf{D}_3 + \tilde{\mathbf{A}}_3 < 0$	$\alpha \in (1, 1\frac{1}{2}]$
(E.5)	$4\mathbf{C}_1 + \tilde{\mathbf{A}}_1 + \frac{3}{4}(\mathbf{D}_1 + 2\mathbf{E}_1) < 0$	$\alpha \in [1\frac{1}{2}, \alpha^*]$
(E.6)	$4\mathbf{B}_3\tilde{\mathbf{A}}_3 - \mathbf{D}_3^2 > 0$	$\alpha \in [1\frac{1}{2}, \alpha^*]$
(E.7)	$5\frac{4}{9}(\mathbf{C}_3 + \mathbf{F}_3) + \tilde{\mathbf{A}}_3 < 0$	$\alpha \in [1\frac{1}{2}, \alpha^*]$.

Write

$$
\begin{aligned}
g_1(\alpha, x, y) &= \mathbf{P}_1 y^2 + \mathbf{P}_2 y + \mathbf{P}_3 \\
x &= |c_1|^2 \in [0, 4] \\
y &= |c_2| \in [0, 2] \\
\mathbf{P}_1 &= \mathbf{C}_1 \\
\mathbf{P}_2 &= \mathbf{E}_1 x \\
\mathbf{P}_3 &= \tilde{\mathbf{A}}_1 + \mathbf{D}_1 x + \mathbf{B}_1 x^2
\end{aligned}
$$

$$
\begin{aligned}
g_2(\alpha, x, y) &= \mathbf{Q}_1 y^2 + \mathbf{Q}_2 y + \mathbf{Q}_3 \\
x &= |c_1|^2 \in [0, 4] \\
y &= |c_2| \in [0, 2] \\
\mathbf{Q}_1 &= \mathbf{C}_3 + \mathbf{F}_3 \\
\mathbf{Q}_2 &= \mathbf{E}_3 x \\
\mathbf{Q}_3 &= \tilde{\mathbf{A}}_3 + \mathbf{D}_3 x + \mathbf{B}_3 x^2.
\end{aligned}
$$

Proof of (a):
Since $\mathbf{P}_1 < 0$ it follows that

$$\begin{aligned} g_1(\alpha, x, y) &\leq \mathbf{P}_3 - \frac{\mathbf{P}_2^2}{4\mathbf{P}_1} \\ &= \left(\mathbf{B}_1 - \frac{\mathbf{E}_1^2}{4\mathbf{C}_1}\right) x^2 + \mathbf{D}_1 x + \tilde{\mathbf{A}}_1. \end{aligned}$$

Since

$$\left(\mathbf{B}_1 - \frac{\mathbf{E}_1^2}{4\mathbf{C}_1}\right) = \left(1\frac{1}{4}\mu + \frac{3}{8}\alpha\right)^2 \text{ and } -\frac{2\mathbf{D}_1\mathbf{C}_1}{4\mathbf{B}_1\mathbf{C}_1 - \mathbf{E}_1^2} \geq 4 \text{ (E.1)}$$

we find

$$\begin{aligned} g_1(\alpha, x, y) &\leq \frac{9}{16}\left(1\frac{1}{4}\mu + \frac{3}{8}\alpha\right)^2 + \frac{3}{4}\mathbf{D}_1 + \tilde{\mathbf{A}}_1 \\ &< 0 \text{ (E.2)}. \end{aligned}$$

Proof of (b):
Suppose $\alpha \leq 1\frac{1}{2}$. Then $\mathbf{Q}_1 < 0$ (Trivial). Therefore

$$\begin{aligned} g_2(\alpha, x, y) &\leq \mathbf{Q}_3 - \frac{\mathbf{Q}_2^2}{4\mathbf{Q}_1} \\ &= \left(\mathbf{B}_3 - \frac{\mathbf{E}_3^2}{4(\mathbf{C}_3 + \mathbf{F}_3)}\right) x^2 + \mathbf{D}_3 x + \tilde{\mathbf{A}}_3. \end{aligned}$$

Since $\mathbf{B}_3 - \frac{\mathbf{E}_3^2}{4(\mathbf{C}_3+\mathbf{F}_3)} > 0$ (E.3) we find

$$\begin{aligned} g_2(\alpha, x, y) &\leq \max\left(\tilde{\mathbf{A}}_3, \frac{9}{16}\left(\mathbf{B}_3 - \frac{\mathbf{E}_3^2}{4(\mathbf{C}_3 + \mathbf{F}_3)}\right) + \frac{3}{4}\mathbf{D}_3 + \tilde{\mathbf{A}}_3\right) \\ &< 0 \text{ (E.4)}. \end{aligned}$$

Proof of (c):
Let $\alpha \in [1\frac{1}{2}, \alpha^*]$.
We first show that $g_1(\alpha, x, y)$ has no local extrema in G_3:
In such an extremum we have:

$$\begin{cases} \frac{\delta}{\delta x} g_1 = 0 \\ \frac{\delta}{\delta y} g_1 = 0 \end{cases}$$

which implies

$$\begin{cases} x = \frac{-2\mathbf{C}_1\mathbf{D}_1}{4\mathbf{B}_1\mathbf{C}_1 - \mathbf{E}_1^2} \\ y = \frac{\mathbf{D}_1\mathbf{E}_1}{4\mathbf{B}_1\mathbf{C}_1 - \mathbf{E}_1^2}. \end{cases}$$

But, by (E.1), this implies $x > 4$.
Therefore, we only need to consider g_1 on δG_3:

- $x = \frac{3}{4}$, $y \in [r - 1\frac{1}{2}, 2]$.
  This coincides with a part of the proof of (a).
- $x \in [r-2, \frac{3}{4}]$, $y = 2$.
  Note that $\mathbf{B}_1 < 0$. Therefore

$$\begin{aligned} g_1(\alpha, x, y) &= 4\mathbf{C}_1 + \tilde{\mathbf{A}}_1 + (\mathbf{D}_1 + 2\mathbf{E}_1)x + \mathbf{B}_1 x^2 \\ &\leq 4\mathbf{C}_1 + \tilde{\mathbf{A}}_1 + \frac{3}{4}(\mathbf{D}_1 + 2\mathbf{E}_1) \\ &< 0 \text{ (E.5)}. \end{aligned}$$

- $(x, y) \in l$ where $l : 2x + y = r$.
  If $g_1(\alpha, x, y)$ has a local maximum on l, we find that
  - $2\mathbf{P}_1 y dy + \mathbf{E}_1 x dy + \mathbf{E}_1 y dx + \mathbf{D}_1 dx + 2\mathbf{B}_1 x dx = 0$,
  - $2dx + dy = 0$,
  - $2x + y = r$.

  This implies $\begin{cases} x &= \frac{x_n}{x_d} \\ y &= \frac{y_n}{y_d} \end{cases}$
  where

$$\begin{aligned} x_n &= -\mathbf{D}_1 - r(\mathbf{E}_1 - 4\mathbf{C}_1) \\ x_d &= 2\mathbf{B}_1 - 4\mathbf{E}_1 + 8\mathbf{C}_1 \\ y_n &= 2\mathbf{D}_1 + 2r(\mathbf{B}_1 - \mathbf{E}_1) \\ y_d &= x_d. \end{aligned}$$

  Our claim that $g_1(\alpha, x, y) < 0$ now follows if

$$\mathbf{C}_1 y_n^2 + \mathbf{E}_1 x_n y_n + \tilde{\mathbf{A}}_1 x_d^2 + \mathbf{D}_1 x_n x_d + \mathbf{B}_1 x_n^2 < 0.$$

  Note that $y_n = r x_d - 2x_n$. A brief calculation now shows that the previous inequality is equivalent to

$$x_d \left((r^2 \mathbf{C}_1 + \tilde{\mathbf{A}}_1) x_d - \frac{1}{2} x_n^2 \right) < 0^{\dagger}.$$

Proof of (d):
This case is similar to the previous. Let $\alpha \in [1\frac{1}{2}, \alpha^*]$.
If $g_2(\alpha, x, y)$ has a local extremum in G_4 we find that

$$\begin{cases} x = \frac{-2(\mathbf{C}_3 + \mathbf{F}_3)\mathbf{D}_3}{4\mathbf{B}_3(\mathbf{C}_3 + \mathbf{F}_3) - \mathbf{E}_3^2} \\ y = \frac{\mathbf{D}_3 \mathbf{E}_3}{4\mathbf{B}_3(\mathbf{C}_3 + \mathbf{F}_3) - \mathbf{E}_3^2}. \end{cases}$$

Since $\mathbf{D}_3 \geq 0, \mathbf{E}_3 \geq 0$, but $4\mathbf{B}_3(\mathbf{C}_3 + \mathbf{F}_3) - \mathbf{E}_3^2 < 0$ (E.3) it follows that $y \leq 0$. Therefore, we only need to consider g_2 on δG_4:

- $y = 0,\ x \le \frac{r}{2}$.
  Note that $\mathbf{B}_3 < 0$. Therefore

$$\begin{aligned} g_2(\alpha, x, 0) &= \tilde{\mathbf{A}}_3 + \mathbf{D}_3 x + \mathbf{B}_3 x^2 \\ &\le \tilde{\mathbf{A}}_3 - \frac{\mathbf{D}_3^2}{4\mathbf{B}_3} \\ &< 0 \text{ (E.6)}. \end{aligned}$$

- $x = 0,\ y \in [0, r]$.
  Then

$$\begin{aligned} g_2(\alpha, 0, y) &= (\mathbf{C}_3 + \mathbf{F}_3) y^2 + \tilde{\mathbf{A}}_3 \\ &\le \max(r^2(\mathbf{C}_3 + \mathbf{F}_3) + \tilde{\mathbf{A}}_3, \tilde{\mathbf{A}}_3) \\ &< 0 \text{ (E.7)}. \end{aligned}$$

- $(x, y) \in l$ where $l : 2x + y = r$.
  As in the previous case we find

$$g_2(\alpha, x, y) \le g_2(\alpha, x^*, y^*).$$

Here $\begin{cases} x^* = \frac{x_n}{x_d} \\ y^* = \frac{y_n}{y_d} \end{cases}$ where

$$\begin{aligned} x_n &= -\mathbf{D}_3 - r(\mathbf{E}_3 - 4(\mathbf{C}_3 + \mathbf{F}_3)) \\ x_d &= 2\mathbf{B}_3 - 4\mathbf{E}_3 + 8(\mathbf{C}_3 + \mathbf{F}_3) \\ y_n &= 2\mathbf{D}_3 + 2r(\mathbf{B}_3 - \mathbf{E}_3) \\ y_d &= x_d. \end{aligned}$$

The claim that $g_2(\alpha, x, y) < 0$ now follows (as in the previous case) since

$$x_d \left(\left(r^2(\mathbf{C}_3 + \mathbf{F}_3) + \tilde{\mathbf{A}}_3 \right) x_d - \frac{1}{2} x_n^2 \right) < 0^\dagger.$$

■

Theorem 2.10 *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbf{B}^*$. *Then*

$$|a_4| \le \frac{2}{e}.$$

We have equality if and only if f is of the form

$$f(z) = \kappa \exp\left(\frac{\xi z^4 - 1}{\xi z^4 + 1} \right), \quad \kappa, \xi \in \Gamma_1.$$

Proof:
From Theorems 2.8 and 2.9 we find

$$|a_4|^2 \leq \nu(\alpha)e^{2\alpha}\alpha^2 = \frac{4}{e^2}$$

thus

$$|a_4| \leq \frac{2}{e}.$$

If we have equality, then $\alpha = -\log|a_0| = 1, a_1 = 0$.
If we write $f = \lambda \exp(-\alpha p)$ with $p = 1 + \sum_{j=1}^{\infty} c_j z^j \in \mathbf{P}, |\lambda| = 1, \alpha = 1$, then $\left|\frac{\alpha^3}{24}c_1^4 - \frac{\alpha^2}{2}c_1^2 c_2 + \alpha c_1 c_3 + \frac{\alpha}{2}c_2^2 - c_4\right|$ reduces to $\left|\frac{1}{2}c_2^2 - c_4\right|$.

Apply (2.10) with $\vec{w} = (0, -\frac{1}{2}c_2, 0, 1)$. We find

$$\left|c_4 - \frac{1}{2}c_2^2\right|^2 \leq 4 - |c_2|^2.$$

Therefore, in the extremal case $c_2 = 0, |c_4| = 2$. By Theorem 1.7, p must be of the form

$$p(z) = \sum_{k=1}^{4} \lambda_k \frac{e^{i\beta + \frac{\pi i k}{2}} + z}{e^{i\beta + \frac{\pi i k}{2}} - z}$$

for some $\beta \in [0, 2\pi]$, $\lambda_k \geq 0$, $(k = 1, \ldots, 4)$ with $\sum_{k=1}^{4} \lambda_k$. Recalculating c_1 and c_2 from this representation yields

$$\begin{aligned} \sum_{k=1}^{4} \lambda_k i^{-k} &= \frac{c_1 e^{i\beta}}{2} = 0, \\ \sum_{k=1}^{4} \lambda_k i^{-2k} &= \frac{c_2 e^{2i\beta}}{2} = 0. \end{aligned}$$

Considering the real and imaginary part of the above expressions we find that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{1}{4}$, i.e.

$$p(z) = \frac{z^4 e^{-4i\beta} - 1}{z^4 e^{-4i\beta} + 1}.$$

From this the theorem follows.

∎

Chapter 3

Univalent Elements of $\mathbf{B}^*$

3.1 Introduction

Define $\mathbf{B}^*_U \subset \mathbf{B}^*$ by
$$\mathbf{B}^*_U = \{f \in \mathbf{B}^* \mid f \text{ is univalent}\}.$$

In this chapter we shall look at the class $\mathbf{B}^*_U$. We shall prove the Hummel-conjecture for $n = 1, 2$ and give an estimate for $\mathcal{A}_3(\mathbf{B}^*_U(\mathbf{R}))$. Note that there are many functions in $\mathbf{B}^*$ which map Δ_1 onto the domain mentioned in Hummel's conjecture.

For $d \in (0,1)$ define $k_d \in \mathbf{B}^*_U$ as the function which maps Δ_1 onto $\{z \mid |z| < 1, z \notin (-1, 0]\}$, with $k_d(0) = d$, $k_d'(0) > 0$. Then the Hummel-conjecture reads:

Conjecture (Hummel, Scheinberg and Zalcman): *For $n \geq 1$: The maximum $\mathcal{A}_n(\mathbf{B}^*_U)$ is attained only for a $k_d \in \mathbf{B}^*_U$ and its rotations.*

3.2 $\mathcal{A}_1(\mathbf{B}^*_U)$

Theorem 3.1 $\mathcal{A}_1(\mathbf{B}^*_U) = 12 - 8\sqrt{2}$.
Equality is attained only by $k_{\sqrt{2}-1}$ and its rotations.

Proof:
Suppose $f \in \mathbf{B}^*_U$, $f(0) = d > 0$.
Let
$$k : z \rightarrow \frac{z}{(1-z)^2}$$
and
$$g = \frac{k \circ f - k(d)}{k'(d) \cdot f'(0)}.$$

Then $g \in S$ and g omits $\frac{-k(d)}{k'(d)\, f'(0)}$.
The Koebe one-quarter-theorem, which states that the range of any $h \in S$ containts the disk $\Delta_{\frac{1}{4}}$, now implies that

$$|f'(0)| \leq \left|\frac{4k(d)}{k'(d)}\right| = \frac{4d(1-d)}{1+d}.$$

The R.H.S. attains its maximum at $d = \sqrt{2} \quad 1$. Therefore $|f'(0)| \leq 12 - 8\sqrt{2}$. We have equality if and only if $|a_1| = \frac{4d(1-d)}{1+d}$ which implies that g is (a rotation of) the Koebe function k. But this implies that f is $k_{\sqrt{2}-1}$ or one of its rotations.

■

3.3 $\mathcal{A}_2(\mathbf{B}_U^*)$

In the following we shall apply a result for Bieberbach-Eilenberg functions. We recall the definition and the result:

Definition: *Let $f \in H(\Delta_1)$. f is called a Bieberbach-Eilenberg function if for all $z_1,\ z_2 \in \Delta_1 :\ f(z_1)f(z_2) \neq 1$.*

Theorem 3.2 *Let f be a univalent Bieberbach-Eilenberg function,*
$f(z) = \sum_{n=0}^{\infty} a_n z^n$. *Then*

$$\left|\frac{a_2}{a_1}\right|^2 \leq -2\log|a_1|.$$

Proof:
See (Pommerenke [23], pp. 101-102, Theorem 4.3).

■

Theorem 3.3 $\mathcal{A}_2(\mathbf{B}_U^*) = 0.45538...$
This value is attained only by k_{d^*} and its rotations, where d^* denotes the smallest positive root of $(1+d)^4 = 12d$.

Before proving the theorem we shall derive some inequalities involving the coefficients of functions in $\mathbf{B}_U^*$.

Lemma 3.1 *Suppose $f \in \mathbf{B}_U^*$, $f(z) = \sum_{n=0}^{\infty} a_n z^n, f(0) = d > 0$. Then*

$$\left|a_2 + \frac{da_1^2}{1-d^2}\right| \leq 2|a_1|\left(1 - \frac{|a_1|}{1-d^2}\right).$$

Proof:
Define

$$g = \frac{f-d}{1-df} \cdot \frac{1-d^2}{f'(0)}.$$

Since $f \in \mathbf{B}_U^*$, it follows that $g \in \mathbf{S}$. Furthermore $|g(z)| \leq \frac{1-d^2}{|f'(0)|}$. This implies that (see Duren [6], p. 74):

$$|g''(0)| \leq 4\left(1 - \frac{|a_1|}{1-d^2}\right).$$

Now note that $g''(0) = 2\left(\frac{da_1}{1-d^2} + \frac{a_2}{a_1}\right)$.

∎

Lemma 3.2 *Suppose* $f \in \mathbf{B}_U^*$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $f(0) = d > 0$. *Then*

$$\left| a_2 - \frac{a_1^2}{d(1-d^2)} \right| \leq 2|a_1|.$$

Proof:
The proof is similar to that of Theorem 3.1.
Let $k : z \to \frac{z}{(1-z)^2}$ and $g = \frac{k \circ f - k(d)}{k'(d)\, f'(0)}$. Then $g \in \mathbf{S}$ and g omits $\frac{-k(d)}{k'(d)\, f'(0)}$. This implies that

$$\left| \frac{g''(0)}{2} - \frac{k'(d) f'(0)}{k(d)} \right| \leq 2.$$

From this the lemma follows.

∎

Lemma 3.3 *Suppose* $f \in \mathbf{B}_U^*$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $f(0) = d > 0$. *Then*

$$\left| a_2 - \frac{1-2d-d^2}{2d(1-d^2)} a_1^2 \right| \leq 2\,|a_1| \sqrt{\frac{1}{2} \log \frac{4d(1-d)}{|a_1|(1+d)}}.$$

Proof:
Since f omits the value 0, the function

$$g : z \to \frac{\sqrt{d}}{1-d} \cdot \frac{1-f(z)}{\sqrt{f(z)}}$$

is holomorphic in Δ_1. Since $|f(z)| < 1 \quad (z \in \Delta_1)$, it follows that g^2 is also univalent. Define

$$h : z \to \frac{g(z)-1}{g(z)+1} = \sum_{j=1}^{\infty} c_j z^j.$$

It is easy to verify that $h(0) = 0$ and for all $z_1, z_2 \in \Delta_1$: $h(z_1)h(z_2) \neq 1$ It follows that h is a univalent Bieberbach-Eilenberg function. From Theorem 3.2 it follows that $|\frac{c_2}{c_1}|^2 \leq -2\log|c_1|$.
Since the power series of g and h are

$$\begin{aligned} g(z) &= 1 + \left(\frac{-(1+d)a_1}{2d(1-d)}\right) z + \left(\frac{-(1+d)a_2}{2d(1-d)} + \frac{d+3}{8d^2(1-d)} a_1^2\right) z^2 + \cdots \\ h(z) &= \left(\frac{-(1+d)a_1}{4d(1-d)}\right) z + \frac{(1+d)}{4d(1-d)} \left[-a_2 + a_1^2 \left(\frac{-d^2 - 2d + 1}{2d(1-d^2)}\right)\right] z^2 + \cdots \end{aligned}$$

the result follows.

■

Proof of Theorem 3.3.

Let $f \subset \mathbf{B}_U^*$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$. The class $\mathbf{B}_U^*$ is rotation-invariant, hence we may assume $f(0) = d > 0$. We have the following constraints on a_2:

$$\left|a_2 + \lambda_i a_1^2\right| \leq 2|a_1|\mu_i \quad (i = 1, 2, 3) \tag{3.1}$$

with

$$\begin{aligned} \lambda_1 &= \tfrac{d}{1-d^2}, & \mu_1 &= 1 - \tfrac{|a_1|}{1-d^2} \\ \lambda_2 &= -\tfrac{1}{d(1-d^2)}, & \mu_2 &= 1 \\ \lambda_3 &= -\tfrac{(1-2d-d^2)}{2d(1-d^2)}, & \mu_3 &= \sqrt{\tfrac{1}{2}\log\tfrac{4d(1-d)}{|a_1|(1+d)}}. \end{aligned}$$

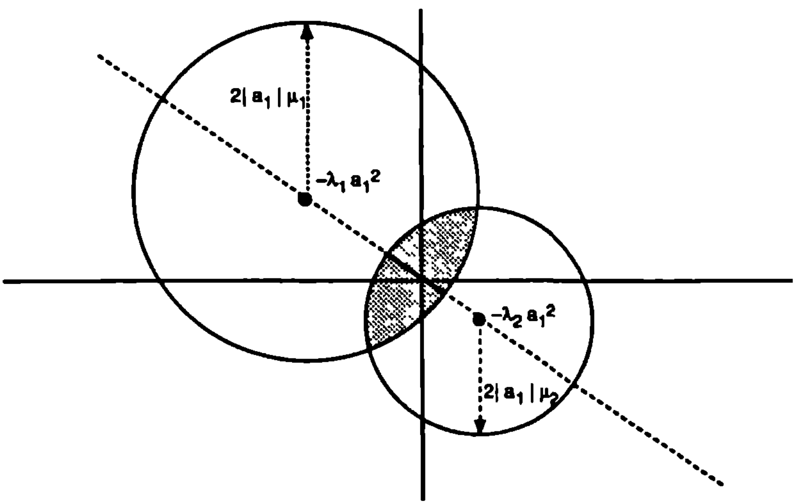

Figure 3.1

(See Figure 3.1). If λ_i and λ_j have opposite sign and p_1 and p_2 denote the intersection points of two circles defined by (3.1), we have

$$|a_2| \leq |p_1| \quad (= |p_2|)$$

i.e.

$$|a_2|^2 \leq \frac{4|a_1|^2}{|\lambda_j - \lambda_i|} \left|\lambda_j \mu_i^2 - \lambda_i \mu_j^2\right| + |\lambda_i||\lambda_j||a_1|^4.$$

It follows that

$$|a_2| \leq \min\{\Phi_1^{\frac{1}{2}}(d,x), \Phi_2^{\frac{1}{2}}(d,x)\}$$

where

$$\begin{aligned}
\Phi_1(d,x) &= \left(\frac{x}{1-d}\right)^2 \left[4d^2 \log \frac{4d(1-d)}{x(1+d)} + p(d,x)\right] \\
\Phi_2(d,x) &= 4x^2 - \frac{8}{1-d^4}x^3 + \frac{3-d^2}{(1-d^2)(1-d^4)}x^4 \\
x &= |a_1| \in \left[0, \frac{4d(1-d)}{1+d}\right] \\
p(d,x) &= (1-2d-d^2)\left\{\frac{7+2d-d^2}{2(1-d^2)^2}x^2 - \frac{8}{1-d^2}x + 4\right\}.
\end{aligned}$$

We shall estimate $\Phi_i^{\frac{1}{2}}(d,x)$ on G_i $(i = 1,2)$ where

$$\begin{aligned}
G_1 &= \left\{(d,x) \mid d \in [0, \sqrt{2}-1],\ x \in \left[0, \frac{4d(1-d)}{1+d}\right]\right\}, \\
G_2 &= \left\{(d,x) \mid d \in [\sqrt{2}-1, 1],\ x \in \left[0, \frac{4d(1-d)}{1+d}\right]\right\}.
\end{aligned}$$

1. $\Phi_1^{\frac{1}{2}}(d,x)$ for $(d,x) \in G_1 \backslash \delta G_1$. Suppose that $\Phi_1(d,x)$ has a local maximum in G_1. Write $y = \frac{x}{1-d^2}$. Then

$$\begin{aligned}
\Phi_1(d,x) &= y^2(1+d)^2 f(d,y) \\
f(d,y) &= 4d^2 \log\left(\frac{4d}{y(1+d)^2}\right) + A(d)y^2 + B(d)y + C(d) \\
A(d) &= \frac{1}{2}(7 - 12d - 12d^2 + d^4) \\
B(d) &= -8(1-2d-d^2) \\
C(d) &= 4(1-2d-d^2).
\end{aligned}$$

In a local maximum (d^*, x^*) we have:

- $\frac{\partial}{\partial d}\Phi_1(d^*, x^*) = 0,$
- $\frac{\partial}{\partial y}\Phi_1(d^*, x^*) = 0,$
- $\frac{\partial^2}{\partial d^2}\Phi_1(d^*, x^*) \le 0.$

Write $y^* = \frac{x^*}{1-d^{*2}}$. We find that

$$(1+2d^*)y^*\frac{\partial}{\partial y}\Phi_1(d^*, y^*) - d^*(1+d^*)\frac{\partial}{\partial d}\Phi_1(d^*, y^*) = 0.$$

Rewrite this to $f_{d^*}(y^*) = 0$ where

$$\begin{aligned} f_d(y) &= A_2(d)y^2 + A_1(d)y + A_0(d) \\ A_2(d) &= d^5 - 24d^3 - 42d^2 + 3d + 14 \\ A_1(d) &= 8(2d^3 + 7d^2 - 3) \\ A_0(d) &= 4(-d^3 - 4d^2 + 2). \end{aligned}$$

Since

$$f_d(\frac{1}{2}) < 0,\ f_d(1) < 0 \text{ and } A_2(d) > 0$$

it follows that $y^* < \frac{1}{2}$.
On the other hand (with $y = \frac{x}{1-d^2}$):

$$\begin{aligned} \frac{\partial^2}{\partial d^2}\Phi_1(d,x) &= y^2\left(2f - 4(1+d)\frac{\partial}{\partial d}f + (1+d)^2\frac{\partial^2}{\partial d^2}f\right) \\ &= y^2\left(-6f + (1+d)^2\frac{\partial^2}{\partial d^2}f\right) \\ &= y^2\left(\log\left(\frac{4d}{y(1+d)^2}\right)\left\{-16d^2 + 16d + 8\right\} + g_d(y)\right) \\ &\ge y^2 g_d(y). \end{aligned}$$

Here

$$\begin{aligned} g_d(y) &= B_2(d)y^2 + B_1(d)y + B_0(d) \\ B_2(d) &= 3(d^4 + 4d^3 + 10d^2 + 4d - 11) \\ B_1(d) &= 32(-d^2 - 2d + 2) \\ B_0(d) &= 4(d^2 + 6d - 5). \end{aligned}$$

Let

$$h_d : y \to 5f_d(y) + 2g_d(y) = C_2(d)y^2 + C_1(d)y + C_0(d)$$

where

$$\begin{aligned}
C_2(d) &= 5d^5 + 6d^4 - 96d^3 - 150d^2 + 39d + 4 \\
C_1(d) &= 8(10d^3 + 27d^2 - 16d + 1) \\
C_0(d) &= 4(-5d^3 - 18d^2 + 12d).
\end{aligned}$$

One easily verifies that

$$h_d(y) > 0 \quad (y \in [0, \frac{1}{2}]).$$

It follows that

$$f_d(y) = 0 \Rightarrow g_d(y) = \frac{h_d(y)}{2} > 0.$$

Therefore, (d^*, x^*) can never be a maximum.

2. $\Phi_1^{\frac{1}{2}}(d, x)$ for $(d, x) \in \delta G_1$.

   (a) $d = \sqrt{2} - 1,\ x \in \left[0, \frac{4d(1-d)}{1+d}\right]$.
   Then $\Phi_1(d, x)$ reduces to

   $$\Phi_1(\sqrt{2} - 1, x) = \left(\frac{x}{1-d}\right) 4d^2 \log \frac{4d(1-d)}{x(1+d)}, \quad d = \sqrt{2} - 1$$

   and from this we find

   $$\Phi_1^{\frac{1}{2}}\left(\sqrt{2} - 1, x\right) \le (12 - 8\sqrt{2})e^{-\frac{1}{2}} \quad (\approx 0.416...).$$

   (b) $d \in [0, \sqrt{2} - 1], x = \frac{4d(1-d)}{1+d}$.
   Then $\Phi_1(d, x)$ reduces to

   $$\Phi_1\left(d, \frac{4d(1-d)}{1+d}\right) = \frac{|1 - 2d - d^2| 8d(1-d)}{(1+d)^3}.$$

   The R.H.S. attains its maximum in case $(1 + d)^4 = 12d$, i.e. $d = d^* \approx 0.141...$. Therefore

   $$\Phi_1^{\frac{1}{2}}\left(d, \frac{4d(1-d)}{1+d}\right) \le \Phi_1^{\frac{1}{2}}\left(d^*, \frac{4d^*(1-d^*)}{1+d^*}\right) \approx 0.45538...$$

   (c) $d \in [0, \sqrt{2} - 1], x = 0$. Then $\Phi_1^{\frac{1}{2}}(d, x) = 0$.

3. $\Phi_2^{\frac{1}{2}}(d,x)$ for $(d,x) \in G_2 \setminus \delta G_2$. To find local extremes we put both

$$\frac{\partial}{\partial d}\Phi_2(d,x) \text{ and } \frac{\partial}{\partial x}\Phi_2(d,x)$$

to zero.

From this we find $x = \frac{8d^2(1-d^2)}{1+4d^2-d^4}$ and then

$$(d^2-1)^2(d^6-5d^4-13d^2+1)=0.$$

However, since we assume that $d \in (\sqrt{2}-1,1)$, this last equation has no roots within this interval.
Therefore, $\Phi_2(d,x)$ has no local extremes in G_2.

4. $\Phi_2^{\frac{1}{2}}(d,x)$ for $(d,x) \in \delta G_2$.

   (a) $d=\sqrt{2}-1, x \in [0, \frac{4d(1-d)}{1+d}]$.
   We can easily solve

   $$\frac{\partial}{\partial x}\Phi_2(\sqrt{2}-1,x)=0.$$

   We find $x = x^* \approx 0.427\ldots$. Therefore

   $$\Phi_2^{\frac{1}{2}}(\sqrt{2}-1,x) \leq \Phi_2^{\frac{1}{2}}(\sqrt{2}-1,x^*) \approx 0.452\ldots$$

   (b) $d \in [\sqrt{2}-1,1], x = \frac{4d(1-d)}{1+d}$.
   Then $\Phi_2(d,x)$ reduces also to

   $$\Phi_2\left(d, \frac{4d(1-d)}{1+d}\right) = \frac{|1-2d-d^2|8d(1-d)}{(1+d)^3}.$$

   The same argument as in case 2(b) now holds. We find a local maximum for $d \approx 0.706\ldots$ with value $\approx 0.304\ldots$.

   (c) $d \in [0,\sqrt{2}-1], x=0$. Then $\Phi_2^{\frac{1}{2}}(d,x)=0$.

From the foregoing we conclude that $|a_2|$ is maximized only if $|a_1| = \frac{4d^*(1-d^*)}{1+d^*}$. Note that in this case the previous inequalities become equalities. This immediately shows the sharpness of the result. The fact $|a_1| = \frac{4d^*(1-d^*)}{1+d^*}$ now implies that the extremal function is indeed the slit mapping described in the theorem.

3.4 $\mathcal{A}_3(\mathbf{B}_U^*(\mathbf{R}))$

Let $\mathbf{B}_U^*(\mathbf{R}) \subset \mathbf{B}_U^*$ denote the subclass of $\mathbf{B}_U^*$ containing those functions whose coefficients are real. We have just seen that for $n = 1, 2$:

$$\mathcal{A}_n(\mathbf{B}_U^*(\mathbf{R})) = \mathcal{A}_n(\mathbf{B}_U^*).$$

Unfortunately, for larger n we only have the result

$$\mathcal{A}_n(\mathbf{B}_U^*(\mathbf{R})) \leq \mathcal{A}_n(\mathbf{B}_U^*) \leq \frac{1}{n}\sqrt{n}$$

which follows from Rogosinski's Theorem 1.3.

In this section we present an estimate for the coefficient $\mathcal{A}_3(\mathbf{B}_U^*(\mathbf{R}))$.

First some preliminaries:

Theorem 3.4 *Let $b \in (0,1)$, $\beta \in (0,b)$. Let $\theta(u) \in C[b,1]$. Then*

$$j_{min}(b,\beta) \leq \left(\int_b^1 \cos\theta(u)du + \beta\right)^2 - \int_b^1 u\cos^2\theta(u)du \leq j_{max}(b,\beta)$$

where

$$\begin{aligned}
j_{min}(b,\beta) &= (b-1+\beta)^2 - \frac{1}{2}(1-b^2)\\
j_{max}(b,\beta) &= \begin{cases} \frac{1}{2}(\sigma-b)^2 + \sigma\beta & \text{if } \beta \geq b + b\log b\\ \frac{\beta^2}{1+\log b} & \text{if } \beta < b + b\log b.\end{cases}
\end{aligned}$$

Here σ denotes the largest root of $\sigma\log\sigma = \beta - b$. All bounds are sharp.

Proof:
Given $\theta(u)$, we consider a variation $\tilde{\theta}(u) = \theta(u) + \epsilon(u)$, where $\epsilon(u) \in C[b,1]$. The amount of variation is:

$$\begin{aligned}
&\int_b^1 \epsilon(u)\tau_1(\beta,u)du + \int_b^1 \epsilon^2(u)\tau_2(\beta,u)du \qquad (3.2)\\
+ &\left(\int_b^1 -\epsilon(u)\sin\theta(u)du\right)^2 + \mathcal{O}(\epsilon^3(u))
\end{aligned}$$

where

$$\begin{aligned}
\tau_1(\beta,b) &= -2\left(\int_b^1 \cos\theta(u)du + \beta\right)\sin\theta(u) + u\sin 2\theta(u),\\
\tau_2(\beta,b) &= \cos 2\theta(u) - \left[\int_b^1 \cos\theta(u)du + \beta\right]\cos\theta(u).
\end{aligned}$$

From the first order variation we see that we may assume

$$\sin\theta(u)\left[u\cos\theta(u)-\left(\int_b^1\cos\theta(u)du+\beta\right)\right]=0, \tag{3.3}$$

since otherwise, the function $\tilde{\theta}(u)$, for suitable choice of $\epsilon(u)$, has better minimizing or maximizing properties. We shall now further assume (3.3).

1. Maximalization of

$$\left(\int_b^1\cos\theta(u)du+\beta\right)^2-\int_b^1 u\cos^2\theta(u)du. \tag{3.4}$$

The expression (3.4) will not decrease if we replace $\cos\theta(u)$ by $|\cos\theta(u)|$. Therefore in the maximizing case we may assume $\cos\theta(u)\geq 0$, $(u\in[b,1])$. Then from (3.3) it follows that $\cos\theta(u)$ has one of the three following forms (see Figure 3.2):

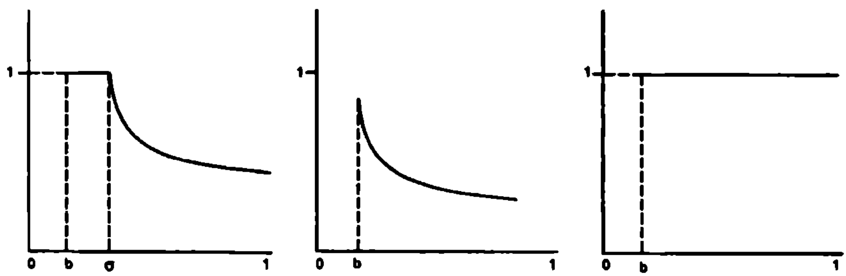

Figure 3.2

(a) Case 1: $\cos\theta(u)=\begin{cases}1 & \text{if } u\in[b,\sigma]\\ \frac{\sigma}{u} & \text{if } u\in[\sigma,1]\end{cases}$
for some $\sigma\in[b,1]$.
Then

$$\int_b^1\cos\theta(u)du=\sigma-b-\sigma\log\sigma$$

which implies

$$\sigma\log\sigma=\beta-b.$$

It follows that there are at most two possible values for σ. Note that if $b>\exp\left(\frac{\beta}{b}-1\right)$, then

$$\sigma\log\sigma\geq b\log b>\beta-b.$$

Therefore, case 1 can only occur if $b \le \exp\left(\frac{\beta}{b} - 1\right)$, i.e.
$\beta \ge b + b\log b$. Then, if σ denotes the largest root of $\sigma\log\sigma = \beta - b$ the value corresponding to (3.4) is

$$\frac{1}{2}(\sigma - b)^2 + \sigma\beta.$$

(b) Case 2: $\cos\theta(u) = \frac{\sigma}{u}, \quad u \in [b, 1]$.
Then

$$\int_b^1 \cos\theta(u)du = -\sigma\log b$$

which implies

$$\sigma = \frac{\beta}{1 + \log b}.$$

Since $\sigma \le b$, this is only possible if $b \ge \exp\left(\frac{\beta}{b} - 1\right)$. The corresponding value of (3.4) is

$$\frac{\beta^2}{1 + \log b}.$$

(c) Case 3: $\cos\theta(u) = 1, \quad u \in [b, 1]$.
Consider, again, the variation (3.2). Since we have already assumed (3.3), it follows that for small $\epsilon(u) \in C[b, 1]$,

$$-(1 - b + \beta)\int_b^1 \epsilon^2(u)du + \int_b^1 \epsilon^2(u)udu \le 0$$

i.e.

$$\int_b^1 \epsilon^2(u)\{-1 + b - \beta + u\}\,du \le 0.$$

However, if we choose $\epsilon(u)$ such that all mass of $\epsilon(u)$ lies around $t \in (b, 1)$ we find that

$$-1 + b - \beta + t \le 0 \quad (t \in (b, 1)).$$

Now let $t \to 1$ to obtain a contradiction.
Thus: Case 3 can never be a maximizing case.

The upper bound now follows.

2. Minimalization of

$$\left(\int_b^1 \cos\theta(u)du + \beta\right)^2 - \int_b^1 u\cos^2\theta(u)du. \tag{3.5}$$

Note that under the condition (3.3), we can never have

$$\int_b^1 \cos\theta(u)du + \beta = 0,$$

unless $\theta(u)$ is constant. From this, it follows that in the minimizing case $\cos\theta(u) < 0$. There, again, are three possibilities:

(a) Case 1: $\cos\theta(u) = \begin{cases} -1 & \text{if } u \in [b,\sigma] \\ \frac{-\sigma}{u} & \text{if } u \in [\sigma, 1]. \end{cases}$

(b) Case 2: $\cos\theta(u) = \frac{-\sigma}{u}, \quad u \in [b, 1].$

(c) Case 3: $\cos\theta(u) = -1, \quad u \in [b, 1].$

Now note that if we can rearrange $\cos\theta(u)$ (with $\theta(u)$ remaining continuous) such that $\int_b^1 \cos\theta(u)du$ does *not* change, but $\int_1^b u\cos^2\theta(u)du$ increases, the corresponding $\cos\theta(u)$ can never minimize (3.5) (see Figure 3.3).

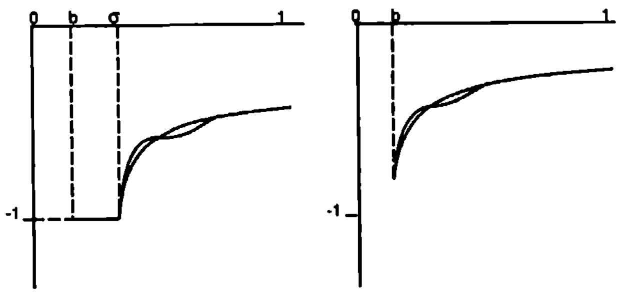

Figure 3.3

It follows that $\cos\theta(u)$ must be non-increasing. Therefore, case 3 minimalizes (3.5). This yields a value

$$(b - 1 + \beta)^2 - \frac{1}{2}(1 - b^2). \tag{3.6}$$

■

Let $b \in (0,1)$. Define $\mathbf{S}_{\mathbf{R}}(b) \subset H(\Delta_1)$ by

$$\mathbf{S}_{\mathbf{R}}(b) = \{ \; f \in \mathbf{B} \mid f(0) = 0, f'(0) = b, \; f \text{ is univalent and has real coefficients}\}.$$

We are interested in the coefficients of functions in $\mathbf{S}_{\mathbf{R}}(b)$. Tammi [29] developed a method to obtain information about these coefficients by proving an analogue of the Loewner differential equation and noting that the subclass of $\mathbf{S}_{\mathbf{R}}(b)$, consisting of functions, whose image is a symmetric two-slitted disc, is dense.

Theorem 3.5 (Tammi) *Let D_l be a symmetric two-slitted disc with $0 \in D_l$. Let $f_{D_l} \in \mathbf{S_R}(b)$ be the map from Δ_1 to D_l. There exists a family of mappings $f(\ ,u)$ $(u \in [b,1])$ such that for every $u \in [b,1]$, $f(\ ,u)$ is an element of* $\mathbf{B}$ *and such that*

- *f satisfies the differential equation*

$$u\frac{\partial}{\partial u}f = \frac{f - f^3}{1 - 2f\cos(\theta(u)) + f^2} \tag{3.7}$$

  where $\theta(u)$ is a real-valued continuous function.
- *$f(z,1) = z$.*
- *$f(z,b) = f_{D_l}$.*

■

Theorem 3.6 *Let $b \in (0,1)$, $\phi \in \mathbf{S_R}(b)$, $\gamma \in (0,b)$, $\phi(z) = b(z + A_2z^2 + \cdots$. Then*

$$h_{min}(b,\gamma) \le A_3 - \gamma A_2 \le h_{max}(b,\gamma)$$

where

$$\begin{aligned} h_{min}(b,\gamma) &= 4j_{min}\left(b, \frac{\gamma}{4}\right) + 1 - b^2 - \frac{\gamma^2}{4} \\ h_{max}(b,\gamma) &= 4j_{max}\left(b, \frac{\gamma}{4}\right) + 1 - b^2 - \frac{\gamma^2}{4}. \end{aligned}$$

All bounds are sharp.

Proof:
From the foregoing theorem it follows that we may restrict ourselves to functions in $\mathbf{S_R}(b)$ generated by some $\theta(u) \in C[b,1]$ and the differential equation (3.7). Let $\phi(z) \in \mathbf{S_R}(b)$ be generated by some $\theta(u)$. Substituting

$$\phi_u(z) = b(u)\left(z + A_2(u)z^2 + A_3(u)z^3 + \cdots\right)$$

in the differential equation and comparing coefficients we find:

$$\begin{aligned} u\frac{\partial}{\partial u}b(u) &= b(u) \\ u\frac{\partial}{\partial u}(b(u)A_2(u)) &= 2b^2(u)\cos\theta(u) + b(u)A_2(u) \\ u\frac{\partial}{\partial u}(b(u)A_3(u)) &= b(u)A_3(u) - 2b^3(u) + \\ &\quad 4b^2(u)\cos\theta(u)A_2(u) + 4\cos^2\theta(u)b^3(u). \end{aligned}$$

With the initial conditions $b(1) = 1, A_2(1) = 0, A_3(1) = 0$, this leads to

$$\begin{aligned} b(u) &= u \\ A_2'(u) &= 2\cos\theta(u) \\ A_3'(u) &= 4\cos\theta(u)A_2(u) - 2u + 4\cos^2\theta(u). \end{aligned}$$

It follows that

$$\begin{aligned} A_2 = A_2(b) &= -2\int_b^1 \cos\theta(u)du \\ A_3 = A_3(b) &= -\int_b^1 4\cos\theta(u)\left(-2\int_u^1 \cos\theta(u)dt\right)du \\ &\quad -\int_b^1 4\cos^2\theta(u)du + \int_b^1 2udu \\ &= 4\left(\int_b^1 \cos\theta(u)du\right)^2 - 4\int_b^1 u\cos^2\theta(u)du + 1 - b^2 \\ &= A_2^2 - 2\int_b^1 u\cos 2\theta(u)du. \end{aligned}$$

Then

$$A_3 - \gamma A_2 = 4\left(\int_b^1 \cos\theta(u)du + \frac{\gamma}{4}\right)^2 - 4\int_b^1 u\cos^2\theta(u)du + 1 - b^2 - \frac{\gamma^2}{4}.$$

The theorem now follows from Theorem 3.4.

■

We now turn to the theorem announced in the beginning of this section.

Theorem 3.7 *Let* $\phi \in \mathbf{B}_U^*(\mathbf{R})$, $\phi(z) = \sum_{n=0}^{\infty} a_n z^n, \phi(0) = a > 0$, $\phi'(0) = a_1 > 0$. *Then*

$$k_{min}(a) \le a_3 \le k_{max}(a)$$

where

$$\begin{aligned} k_{min}(a) &= \begin{cases} 0 & a \le \alpha_1 \\ g_{min}\left(a, \frac{4a}{(1+a)^2}\right) & a \in [\alpha_1, \alpha_2] \\ g_{min}\left(a, \frac{4(2+a)+\sqrt{7(a+1)(a+3)}}{3(5+4a+a^2)}\right) & a \in [\alpha_2, 1] \end{cases} \\ k_{max}(a) &= \max_{b \in \left[0, \frac{4a}{(1+a)^2}\right]} g_{max}(a, b) \\ g_{min}(a, b) &= b(1-a^2)\left((2b-2+ab)^2 + b^2 - 1\right) \\ g_{max}(a, b) &= b(1-a^2)\left(h_{max}(2ab, b) + a^2b^2\right). \end{aligned}$$

Here $\alpha_1 (\approx 0.185\ldots)$ *denotes a root of* $-3a^4 - 20a^3 + 18a^2 + 12a + 3 = 0$.
$\alpha_2 (\approx 0.34\ldots)$ *denotes a root of* $19a^4 + 76a^3 + 98a^2 - 52a + 3 = 0$.
The lower bound is sharp in case $a \le \alpha_2$ *(see Figure 3.4).*

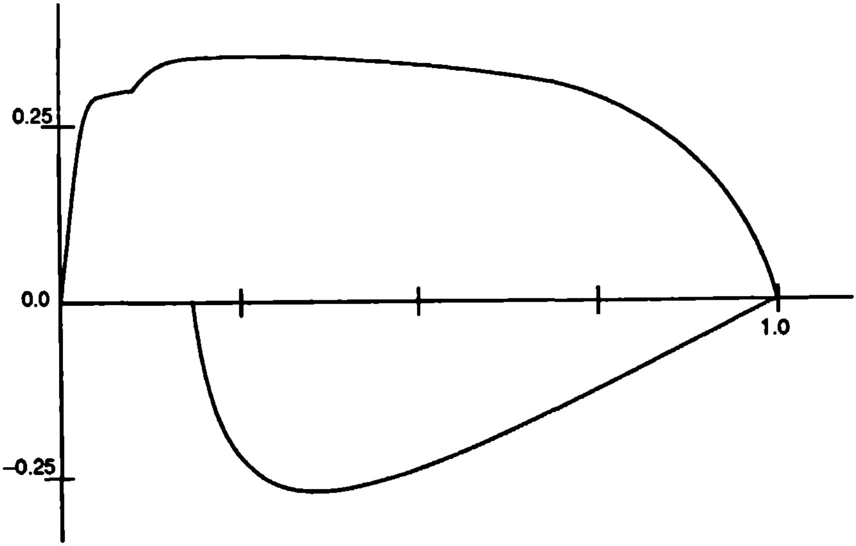

Figure 3.4

Note: The conditions $\phi(0) > 0$, $\phi'(0) > 0$ are not essential: If ϕ is an element of $\mathbf{B}_U^*(\mathbf{R})$, then so are

$$\phi_{\lambda\mu} : z \to \mu\phi(\lambda z), \quad (\lambda, \mu \in \{-1, 1\}).$$

For suitable choices of λ, μ we then have

$$\phi_{\lambda\mu}(0) > 0, \phi'_{\lambda\mu}(0) > 0 \text{ and } a_3(\phi) = \lambda\mu a_3(\phi_{\lambda\mu}).$$

Numerical evaluation of the previous result yields

Corollary 3.1 *Let* $\phi \in \mathbf{B}_U^*(\mathbf{R})$, $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$, $\phi(0) > 0$, $\phi'(0) \ge 0$. *Then*

$$-0.2513... < a_3 < 0.386...$$

However, the functions k_{d_1}, k_{d_2}, $d_1 \approx 0.34\ldots$, $d_2 \approx 0.06\ldots$ *have third coefficient*

$$a_3(k_{d_1}) \approx -0.2510\ldots, \quad a_3(k_{d_2}) \approx 0.315\ldots$$

■

Proof of Theorem 3.7.
Consider

$$\psi : z \to \frac{\phi(z)-a}{1-a\phi(z)} = \frac{a_1}{1-a^2}\left(z + A_2 z^2 + A_3 z^3 + \cdots\right)$$

where

$$\begin{aligned} A_2 &= \frac{a_2}{a_1} + \frac{a a_1}{1-a^2} \\ A_3 &= \frac{a_3}{a_1} + \frac{2 a a_2}{1-a^2} + \frac{a_1^2 a^2}{(1-a^2)^2}. \end{aligned}$$

Then $\psi \in \mathbf{S_R}(b), b = \frac{a_1}{1-a^2}$ and

$$a_3 = b(1-a^2)\left(A_3 - 2abA_2 + a^2b^2\right).$$

From Theorem 3.4 it follows that

$$b(1-a^2)\left(h_{min}(b,2ab) + a^2b^2\right) \leq a_3 < b(1-a^2)\left(h_{max}(b,2ab) + a^2b^2\right). \quad (3.8)$$

- The lower bound.
  To obtain a lower bound we minimize

  $$b(1-a^2)\left((2b-2+ab)^2 + b^2 - 1\right) \quad (3.9)$$

  over $b \in \left[0, \frac{4a}{(1+a)^2}\right]$.
  Since the R.H.S. is a polynomial in b of degree 3 its minimum can easily be calculated. There are three values to consider, i.e.

  (a) $b = 0$
  (b) $b = \frac{4a}{(1+a)^2}$
  (c) $b = b^*(a)$, where $b^*(a)$ is the largest root of

  $$3(5+4a+a^2)b^2 - 8(2+a)b + 3 = 0.$$

  1. $b^*(a) \geq \frac{4a}{(1+a)^2}$.
     This can only occur if

     $$19a^4 + 76a^3 + 98a^2 - 52a + 3 \leq 0$$

     or

     $$\frac{4a}{(1+a)^2} \leq \frac{8(2+a)}{6(5+4a+a^2)}$$

     i.e.

     $$a \in [0.06..., \alpha_2] \cup [0, 0.18...] = [0, \alpha_2].$$

We only have to compare the values for $b = 0$ and $b = \frac{4a}{(1+a)^2}$. It follows that if

$$3 + 12a + 18a^2 - 20a^3 - 3a^4 \leq 0,$$

i.e. $a \in [\alpha_1, \alpha_2]$, then $b = \frac{4a}{(1+a)^2}$ minimizes (3.9), otherwise $b = 0$ minimizes (3.9).

2. $b^*(a) \leq \frac{4a}{(1+a)^2}$.
This can only occur if

$$19a^4 + 76a^3 + 98a^2 - 52a + 3 \geq 0$$

and

$$\frac{4a}{(1+a)^2} \geq \frac{8(2+a)}{6(5+4a+a^2)},$$

i.e. $a \in [\alpha_2, 1]$. It is easy to show that for all a the value in $b^*(a)$ is negative. It follows that in this case $b = b^*(a)$ minimizes (3.9).

- The upper bound.
To obtain an upper bound we have to maximize

$$b(1-a^2)\left(h_{max}(b, 2ab) + a^2b^2\right) \tag{3.10}$$

over $b \in [0, \frac{4a}{(1+a)^2}]$. However, since we know the bound we can obtain using this method is *not* sharp, we confine ourselves to note that we should compare the values

$$\max_{b\in\left[0,\frac{4a}{(1+a)^2}\right]\cap\left[0,\exp(\frac{a}{2}-1)\right]} b(1-a^2)\left(2\sigma^2 + 2\sigma b(a-2) + 1 + b^2\right) \tag{3.11}$$

and

$$\max_{b\in\left[0,\frac{4a}{(1+a)^2}\right]\cap\left[\exp(\frac{a}{2}-1),1\right]} b(1-a^2)\left(\frac{a^2b^2}{1+\log b} + 1 - b^2\right) \tag{3.12}$$

where σ denotes the largest root of $\sigma \log \sigma = b(\frac{a}{2} - 1)$. From Figure 3.5 we see that if $a \leq 0.139...$ then (3.11) maximizes the expression (3.10), otherwise (3.12).

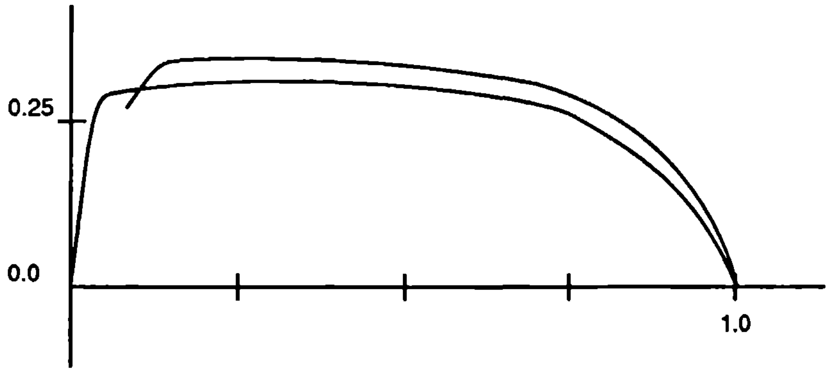

Figure 3.5

Let us consider in which case the lower bound is sharp.

1. $a \in [0, \alpha_1]$.
   The lowerbound, 0, is trivially sharp.

2. $a \in [\alpha_1, 1]$.
   Since $\psi = \frac{\phi - a}{1 - a\phi}$, the extremal ψ from Theorem 3.7 should omit $-a$.
   We can recalculate ψ from the differential equation (3.7). We know that $\cos\theta(u) = -1,\ u \in [b, 1]$. Therefore

$$\begin{cases} u\frac{\partial}{\partial u}\psi(z,u) = \frac{\psi - \psi^3}{(1+\psi)^2} \\ \psi(z,1) = z. \end{cases}$$

   Thus

$$\frac{\psi}{(1-\psi)^2} = b\frac{z}{(1-z)^2}.$$

   It follows that ψ omits $-a$ if and only if $b = \frac{4a}{(1+a)^2}$. This is the case if $a \in [\alpha_1, \alpha_2]$.

■

Chapter 4

Polynomial Elements of $\mathbf{B}^*$

4.1 Introduction

In this chapter we consider polynomial elements of $\mathbf{B}^*$.

For $m \in \mathbf{N}$ define $\mathbf{P}^*$, $\mathbf{P}^\Gamma$, $\mathbf{P}^*_m$, $\mathbf{P}^\Gamma_m \subset \mathbf{B}^*$ by

$$\begin{aligned}
\mathbf{P}^* &= \{p \in \mathbf{B}^* \mid p \text{ is a polynomial}\}. \\
\mathbf{P}^\Gamma &= \{p \in \mathbf{P}^* \mid p \text{ has all its zeros on } \Gamma_1\}. \\
\mathbf{P}^*_m &= \{p \in \mathbf{B}^* \mid p \text{ is a polynomial of degree exactly } m\}. \\
\mathbf{P}^\Gamma_m &= \{p \in \mathbf{P}^*_m \mid p \text{ has all its zeros on } \Gamma_1\}.
\end{aligned}$$

Note that, if $p \in \mathbf{P}^*_m \setminus \mathbf{P}^\Gamma_m$, then for all $\tau \in \Gamma_1$ the functions

$$\Lambda_\tau(p) : z \to \frac{p(z) + \tau z^m \overline{p(\frac{1}{\bar{z}})}}{2}$$

lie in $\mathbf{P}^\Gamma_m$ (see Pólya and Szegö [22], Problem 26, p. 88).

It is easy to verify that for all $n \in \mathbf{N} \cup \{0\}$ the sequence

$$m \to \mathcal{A}_n(\mathbf{P}^*_m), \quad m = n, n+1, \dots$$

is non-decreasing. In Theorem 4.1 we shall show that $\mathbf{P}^*$ is a dense subclass of $\mathbf{B}^*$. Therefore, it follows that the above sequence has limit $\mathcal{A}_n(\mathbf{B}^*)$. Note that the sequence

$$m \to \mathcal{A}_n(\mathbf{P}^\Gamma_m), \quad m = n, n+1, \dots$$

is *not* a non-decreasing sequence as can be seen from the facts

$$\begin{aligned}
\mathcal{A}_1(P^\Gamma_1) &= \frac{1}{2}, \\
\mathcal{A}_1(P^\Gamma_3) &= \frac{3}{8}.
\end{aligned}$$

(See Rubinstein [26]).

For $n \in \mathbf{N} \cup \{0\}$ define

$$\begin{aligned} \mathcal{A}_n^*(\mathbf{P}^*) &= \lim_{m \to \infty} \mathcal{A}_n(\mathbf{P}_m^*), \\ \mathcal{A}_n^*(\mathbf{P}^\Gamma) &= \limsup_{m \to \infty} \mathcal{A}_n(\mathbf{P}_m^\Gamma). \end{aligned}$$

Then $\mathcal{A}_n^*(\mathbf{P}^*) = \mathcal{A}_n(\mathbf{P}^*)$ but we shall show that $\mathcal{A}_n^*(\mathbf{P}^\Gamma) < \mathcal{A}_n(\mathbf{P}^\Gamma)$.

Furthermore, we shall consider the subclass of $\mathbf{P}_3^*$ consisting of those functions f which have at least one zero on Γ_1, and have real coefficients.

4.2 $\mathcal{A}_n^*(\mathbf{P}^\Gamma)$

Theorem 4.1 *The class $\mathbf{P}^*$ is a dense subclass of $\mathbf{B}^*$.*

Proof:
From Theorem 1.10 it easily follows that in the class $\mathbf{B}$ the polynomial elements form a dense subclass.

Now let $f \in \mathbf{B}^*, \epsilon > 0, K \subset \Delta_1, K$ compact. We must show that there exists $p \in \mathbf{P}^*$ such that

$$\|f - p\|_K < \epsilon.$$

Let $0 < r < 1$ be such that $K \subset \Delta_r$. Let $r < s < 1$. For $n \in \mathbf{N}$ let p_n be a polynomial element of $\mathbf{B}$ such that $\|f - p_n\|_{\Delta_s} < \frac{1}{n}$. If $n \to \infty$, p_n eventually will have no zero in Δ_s. (Otherwise, f has a zero in $\overline{\Delta_s}$).

Let $n > \frac{2}{\epsilon}$ be such that p_n has no zero in Δ_s. Define

$$q_s : z \to p_n(sz).$$

Then $q_s \in \mathbf{P}^*$. Furthermore

$$\begin{aligned} \|f - q_s\|_K &\leq \sup_{z \in K} |f(z) - f(sz)| + \sup_{z \in K} |f(sz) - p_n(sz)| \\ &\leq (1 - s) \sup_{z \in \overline{\Delta_r}} |f'(z)| + \frac{\epsilon}{2} < \epsilon \end{aligned}$$

if $s \to 1$. Now take $p = q_s$ for some sufficiently large s.

■

Theorem 4.2 *For $n \in \mathbf{N} \cup \{0\}$, $\mathcal{A}_n^*(\mathbf{P}^\Gamma) = \frac{1}{2}\mathcal{A}_n^*(\mathbf{P}^*)$.*

The proof requires a lemma:

Lemma 4.1 (Ankeny-Rivlin) *Let $p \in \mathbf{P}_m^\Gamma$. Then for $z \in \Delta_1$:*

$$|p(z)| \leq \frac{1 + |z|^m}{2}.$$

Proof of Lemma.
A well-known theorem of S. Bernstein (see Duren [6], p. 195) states that if q is a polynomial of degree m, $|q(z)| \leq 1$, $(z \in \Delta_1)$ then

$$|q'(z)| \leq m, \ (z \in \Delta_1).$$

Lax [15] extended this result in case $q \in \mathbf{P}^*$:
Then

$$|q'(z)| \leq \frac{m}{2}.$$

Since the function

$$z \to z^{1-m} q'(z)$$

is analytic for $1 \leq |z| \leq \infty$, it follows from the maximum modulus principle that

$$|q'(z)| \leq \frac{m}{2}|z|^{m-1} \quad (|z| \geq 1).$$

Therefore, for $z = re^{i\theta}, r > 1$:

$$|q(z) - q(e^{i\theta})| \leq \int_1^r |q'(se^{i\theta})|\, ds \leq \int_1^r \frac{m}{2} s^{m-1} ds = \frac{r^m - 1}{2}$$

i.e.

$$|q(z)| \leq |q(e^{i\theta})| + \frac{r^m - 1}{2} \leq \frac{r^m + 1}{2}.$$

Now let $q : z \to z^m p(\frac{1}{z})$. Then $q \in \mathbf{P}_m^*$.
Therefore for $|z| \geq 1$:

$$|z^m p(\frac{1}{z})| \leq \frac{|z|^m + 1}{2}$$

i.e. for $|z| \leq 1$:

$$|p(z)| \leq \frac{1 + |z|^m}{2}.$$

■

Proof of Theorem 4.2.
Let $p_1, p_2, \ldots \in \mathbf{P}^*$ be a sequence of polynomials with $p_m \in \mathbf{P}_m^*$, whose n^{th} coefficient, a_n, converges to $\mathcal{A}_n^*(\mathbf{P}^*)$.
For $m > n$ and $\tau \in \Gamma_1$ consider $q_{m\tau} = \Lambda_\tau(p_m)$. Then $q_{m\tau} \in \mathbf{P}^\Gamma$ and its n^{th} coefficient is equal to $\frac{1}{2}(a_n + \tau a_{m-n})$. It follows that there exists a sequence $q_1, q_2, \ldots \in \mathbf{P}^\Gamma$ whose n^{th} coefficient, b_n, satisfies $|b_n| \geq \frac{|a_n|}{2}$. Moreover, degree$(q_m) - m$. Therefore

$$\mathcal{A}_n^*(\mathbf{P}^\Gamma) \geq \frac{1}{2}\mathcal{A}_n^*(\mathbf{P}^*).$$

Now suppose $q_1, q_2, \ldots \in \mathbf{P}^\Gamma$ with degree$(q_m) = m$ is a sequence of polynomials with n^{th} coefficient b_{nm}, for which

$$\limsup_{m\to\infty} b_{nm} = \mathcal{A}_n^*(\mathbf{P}^\Gamma).$$

Let $0 < r < 1$. Define

$$p_m : z \to \frac{q_m(rz)}{\frac{1}{2} + \frac{1}{2}r^m}.$$

Then $p_m \in \mathbf{P}^*$. It follows that

$$\frac{b_n r^n}{\frac{1}{2} + \frac{1}{2}r^m} \leq \mathcal{A}_n^*(\mathbf{P}^*).$$

Let $m \to \infty$. We find

$$2\mathcal{A}_n^*(\mathbf{P}^\Gamma)r^n \leq \mathcal{A}_n^*(\mathbf{P}^*).$$

If $r \to 1$ we see

$$\mathcal{A}_n^*(\mathbf{P}^\Gamma) \leq \frac{1}{2}\mathcal{A}_n^*(\mathbf{P}^*).$$

From this the theorem follows.

∎

As a corollary we can answer part of a question raised by Saff and Sheil-Small [27]:

Corollary 4.1

$$\begin{aligned} & \mathcal{A}_0^*(\mathbf{P}^\Gamma) = \tfrac{1}{2} \\ \textit{For } n = 1,2,3,4: \quad & \mathcal{A}_n^*(\mathbf{P}^\Gamma) = \tfrac{1}{e} \\ \textit{For } n > 4: \quad & \mathcal{A}_n^*(\mathbf{P}^\Gamma) \leq \tfrac{C(B^*)}{2} < \tfrac{1}{2} \end{aligned}$$

where $C(B^*)$ is as defined in section 2.3.

∎

4.3 $\mathcal{A}_n(\mathbf{P}^\Gamma)$

In [27], Saff and Sheil-Small give an upperbound for $\mathcal{A}_n^*(\mathbf{P}_m^\Gamma)$:

Theorem 4.3 (Saff and Sheil-Small) *Let $m \in \mathbf{N}$, $0 \leq n \leq m$, $2n \neq m$. Then*

$$\mathcal{A}_n(\mathbf{P}_m^\Gamma) \leq \frac{1}{2}.$$

We have equality if and only for $z \to \frac{\lambda + \mu z^m}{2}$, $|\lambda| = 1, |\mu| = 1$ and $n = 0$ or $n = m$.

The proof is very elegant, therefore we present it here. It will follow easily from the following lemma:

Lemma 4.2 *Let* $p \in \mathbf{P}^{\Gamma}_{m}$, $p(z) = \sum_{j=0}^{m} a_j z^j$. *Then*

$$\sum_{j=0}^{m} |a_j|^2 \leq \frac{1}{2}.$$

Proof of Lemma.
Since all the zeros of p lie on Γ_1, there exists a constant u, $|u| = 1$, such that the coefficients satisfy

$$a_k = u\overline{a_{m-k}}, \quad (k = 0, 1, 2, \dots, m).$$

Therefore $p(z) = \frac{zp'(z)+uq(z)}{m}$, where $q : z \to z^{m-1}\overline{p'(\frac{1}{\bar{z}})}$. Define

$$w : z \to \frac{zp'(z)}{uq(z)}.$$

Then $p(z) = \frac{uq(z)}{m}(1 + w(z))$. Now note that

- $w(z)$ is analytic in Δ_1,
- $|q(z)| = |p'(z)| \leq \frac{m}{2}$ for $z \in \Gamma_1$.

The first claim follows from the Gauss-Lucas theorem, which states that the zeros of p' lie in the convex hull of the set of zeros of p (see Marden [18]).

The second claim follows from the theorem by Lax. (See Lemma 4.1).

It follows that for $z \in \Gamma_1$

$$|p(z)| \leq \frac{1}{2}|1 + w(z)|.$$

Since w is analytic, $w(0) = 0, |w(z)| = 1$ for $z \in \Gamma_1$, $1 + w$ is subordinate to $1 + z$ in Δ_1. Therefore

$$\begin{aligned}\sum_{k=0}^{m} |a_k|^2 &= \frac{1}{2\pi}\int_0^{2\pi} |p(e^{i\theta})|^2 d\theta \\ &\leq \frac{1}{4}\frac{1}{2\pi}\int_0^{2\pi} |1 + w(e^{i\theta})|^2 d\theta \\ &\leq \frac{1}{4}\frac{1}{2\pi}\int_0^{2\pi} |1 + e^{i\theta}|^2 d\theta = \frac{1}{2}.\end{aligned}$$

■

Proof of Theorem 4.3.

Let $p \in \mathbf{P}^{\Gamma}_m$, $p(z) = \sum_{k=0}^{m} a_k z^k$. Note that $|a_n| = |a_{m-n}|$. Now, if $2n \neq m$, it follows from the lemma that

$$2|a_n|^2 \leq \frac{1}{2}$$

which implies $|a_n| \leq \frac{1}{2}$.

If $|a_n| = \frac{1}{2}$, then $p(z) = \frac{\xi z^n + \mu z^{m-n}}{2}$ with $|\xi| = 1, |\mu| = 1$. But then $p(0) = 0$, unless $n = 0$ or $n = m$.

∎

Note: The missing coefficient problem, the case $2n = m$, was later solved by Kristiansen [12], as a result of his theorem on trigonometric polynomials with only real roots:

In this case the result $\mathcal{A}_n(\mathbf{P}^{\Gamma}_m) \leq \frac{1}{2}$ is also true; The functions

$$z \to \mu \left(\frac{1 + \lambda z^n}{2}\right)^2, \quad |\lambda| = 1, \ |\mu| = 1$$

are extremal.

Remark: From the foregoing we conclude that for all $n \in \mathbf{N} \cup \{0\}$:

$$\mathcal{A}^*_n(\mathbf{P}^{\Gamma}) < \mathcal{A}_n(\mathbf{P}^{\Gamma}).$$

An easy consequence of the foregoing is the following

Corollary 4.2 *Let $m \in \mathbf{N}$, $\frac{m}{2} \leq n \leq m$. Then $\mathcal{A}_n(\mathbf{P}^*_m) = \frac{1}{2}$.*

Proof:

Let $p \in \mathbf{P}^*_m$, $p(z) = \sum_{j=0}^{m} a_j z^j$. Consider

$$q_\tau : z \to \frac{p(z) + \tau z^{2n-m} z^m \overline{p}(\frac{1}{z})}{2}.$$

Then $q_\tau \in \mathbf{P}^{\Gamma}_m$ for all $\tau \in \Gamma_1$ except for at most one value. Its n^{th} coefficient is

$$\frac{a_n + \tau \overline{a_n}}{2}.$$

This implies that $|a_n| \leq \frac{1}{2}$, thus $\mathcal{A}_n(\mathbf{P}^*_m) \leq \frac{1}{2}$.

Let $\epsilon > 0$ and consider $p(z) = \frac{1}{2} + (\frac{1}{2} - \epsilon)z^n + \epsilon z^m$. Then $p(z) \in \mathbf{P}^*_m$ and therefore $\mathcal{A}_n(\mathbf{P}^*_m) \geq \frac{1}{2} - \epsilon$ $(\epsilon > 0)$. The result now follows.

∎

Theorem 4.4 *Let $m \in \mathbf{N}, 1 \leq n \leq m$.*
There exists $p \in \cup_{j=1}^m \mathbf{P}_j^*$, $p(z) = \sum_{j=1}^{m} a_j z^j$ such that

$$|a_n| = \mathcal{A}_n(\mathbf{P}_m^*).$$

Moreover $0 \in \overline{p(\Delta_1)}$.

Proof:
Note that $\cup_{j=1}^m \mathbf{P}_j^* \cup \{$ p is constant $, |p(z)| \leq 1\}$ is compact. Therefore (see Theorem 1.1) the maximum is attained. From Rouché's theorem it follows that $0 \in \overline{p(\Delta_1)}$.

■

4.4 $\mathcal{A}_0(\delta\mathbf{P}_m^*)$

Define $\delta\mathbf{P}_m^* = \left\{p \in \cup_{j=1}^m \mathbf{P}_j^* \mid 0 \in \overline{p(\Delta_1)}\right\}$. Then for $1 \leq n \leq m$:

$$\mathcal{A}_n(\delta\mathbf{P}_m^*) = \mathcal{A}_n(\mathbf{P}_m^*).$$

Of course, $\mathcal{A}_0(\mathbf{P}_m^*) = 1$, but what can we say about $\mathcal{A}_0(\delta\mathbf{P}_m^*)$?

Theorem 4.5 $\mathcal{A}_0(\delta\mathbf{P}_2^*) = \frac{3}{8}\sqrt{3}$.
Equality is attained by $p : z \to \frac{3}{8}\sqrt{3}(1-z)(1+\frac{1}{3}z)$.

Proof:
Let $p \in \delta\mathbf{P}_2^*$ maximize $\mathcal{A}_0(\delta\mathbf{P}_2^*)$. Then degree$(p) = 2$. Since $\delta\mathbf{P}_2^*$ is rotation-invariant we may assume $p(1) = 0$.

Let $q : z \to z^2 p(\frac{1}{z})$. Then $q(1) = 0$, $q(\alpha) = 0$ for some $\alpha \in \overline{\Delta_1}$. Furthermore, q has leading coefficient $C = \mathcal{A}_0(\delta\mathbf{P}_2^*)$. We can write

$$q(z) = C(z-\alpha)(z-1)$$

where $C \leq \dfrac{1}{\max\limits_{|z|=1} |(z-\alpha)(z-1)|}$.
It follows that

$$\mathcal{A}_0(\delta\mathbf{P}_2^*) = \max_{\alpha \in \overline{\Delta_1}} \min_{|z|=1} \frac{1}{|(z-\alpha)(z-1)|}.$$

Note that

$$\max_{|z|=1} |(z - \Re e\, \alpha)(z-1)| \leq \max_{|z|=1} |(z-\alpha)(z-1)| \quad (\alpha \in \overline{\Delta_1}).$$

Therefore we may assume $\alpha \in (-1,1)$. Then, routine calculations yield

$$\max_{|z|=1} |(z-\alpha)(z-1)| = \begin{cases} 2(1+\alpha) & \alpha \geq 2\sqrt{2}-3 \\ \frac{(1+|\alpha|)^2}{2\sqrt{|\alpha|}} & \alpha \leq 2\sqrt{2}-3. \end{cases}$$

It follows that the optimal choice for α is $\alpha = -\frac{1}{3}$. Thus

$$|a_0| \leq \frac{\frac{2}{3}\sqrt{3}}{(\frac{4}{3})^2} = \frac{3}{8}\sqrt{3}.$$

We have equality for the function as proposed in the theorem. From this the theorem follows.

■

Theorem 4.6

$$\lim_{m\to\infty} \mathcal{A}_0(\delta\mathbf{P}^*_m) = 1.$$

Proof:
Let $\alpha \in (0,1)$ and consider

$$f_\alpha : z \to \frac{1}{2} + \frac{1}{2}\left(\frac{z+\alpha}{1+\alpha z}\right).$$

Then $f_\alpha \in \mathbf{B}^*$. Therefore there exists a sequence $p_1, p_2, \ldots \in \mathbf{P}^*$ which converges to f_α. Then $p_j(0)$ converges to $f(0) = \frac{1}{2}(1+\alpha)$, i.e.

$$|p_j(0)| \geq \frac{1}{2}(1+\alpha) - \delta_j$$

with $\delta_j \to 0 \quad (j \to \infty)$. Furthermore, for $0 < r < 1$ we have

$$\min_{|z|=r} |p_j(z)| \to \min_{|z|=r} |f(z)| = \frac{1}{2}(1-r)\left(\frac{1+\alpha}{1-r\alpha}\right).$$

Therefore

$$\min_{|z|=1} |p_j(z)| < \frac{1}{2}(1-r)\left(\frac{1+\alpha}{1-r\alpha}\right) \quad (j \to \infty).$$

Let

$$\epsilon_j = \min_{|z|=1} |p_j(z)| = \kappa_j p_j(e^{i\theta_j})$$

for some κ_j, θ_j with $|\kappa_j| = 1,\ \theta_j \in [0, 2\pi]$. Define

$$q_j : z \to \frac{p_j(z) - \rho_j z}{1+\epsilon_j}$$

where $\rho_j = -\epsilon_j \overline{\kappa_j} e^{-i\theta_j}$. Then, according to Rouché's Theorem, $q_j \in \mathbf{P}^*_m$, for some $m \in \mathbf{N}$. Therefore

$$\mathcal{A}_0(\delta\mathbf{P}^*_m) \geq |q_j(0)| = \left|\frac{p_j(0)}{1+\epsilon_j}\right| \geq \frac{\frac{1}{2}(1+\alpha) - \delta_j}{1 - \frac{1}{2}(1-r)(\frac{1+\alpha}{1-r\alpha})}.$$

Now let $r \to 1, j \to \infty$. Then

$$\liminf_{m\to\infty} \mathcal{A}_0(\delta\mathbf{P}^*_m) \geq \frac{1}{2}(1+\alpha) \quad (\alpha \in (0,1)).$$

Now let $\alpha \to 1$ to obtain the desired result.

■

4.5 $\mathcal{A}_n(\delta\mathbf{P}_3^*(\mathbf{R}))$ $(n = 0, 1, 2, 3)$

Let $\mathbf{P}_m^*(\mathbf{R}) \subset \mathbf{P}_m^*$ denote the subclass of $\mathbf{P}_m^*$ containing those functions whose coefficients are real.

Analogously, define $\delta\mathbf{P}_m^*(\mathbf{R})$. We have

$$\mathcal{A}_n(\delta\mathbf{P}_m^*(\mathbf{R})) = \mathcal{A}_n(\delta\mathbf{P}_m^*) \quad (n = 0, 1, \ldots, m)$$

in case $m = 2$. Computer experiments suggest that it is also true for $m = 3$, but we can not prove this, neither for $m = 3$, nor for higher m.

Theorem 4.7

$$\mathcal{A}_0(\delta\mathbf{P}_3^*(\mathbf{R})) = \frac{3 + 2\sqrt{2}}{8}.$$

We have equality for the function

$$p : z \to \frac{\sqrt{2}}{8}(z - 1)(z - 1 - \sqrt{2})(z + 1 + \frac{1}{2}\sqrt{2}).$$

Proof:
From Theorem 4.5 we see that if p has constant term $\mathcal{A}_0(\delta\mathbf{P}_3(\mathbf{R}))$ then the function $q : z \to z^3 p(\frac{1}{z})$ has three zeros, $\alpha_1, \alpha_2, \alpha_3$ in $\overline{\Delta_1}$, which lie symmetric with respect to the real axis, and at least one of them lies on Γ_1. Furthermore, by Theorem 4.3, we may assume that at least one zero lies inside Δ_1.

Note that if p is element of $\delta\mathbf{P}_3(\mathbf{R})$, then so are

$$p_1 : z \to p(-z) \text{ and } p_2 : z \to -p(-z).$$

Therefore there are three cases to consider (see Figure 4.1):

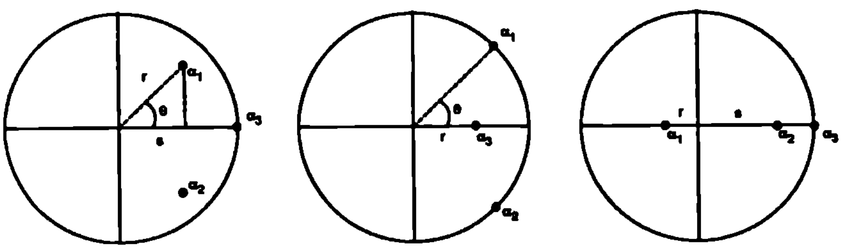

Figure 4.1

1. Case 1: $\alpha_1 = \overline{\alpha_2}$, $\alpha_3 = 1$, $|\alpha_1| < 1$.
   Define $G_1 = \{(\alpha_1, \alpha_2, \alpha_3) \in (\overline{\Delta_1})^3 \mid \alpha_1 = \overline{\alpha_2},\ \alpha_3 = 1,\ |\alpha_1| < 1\}$.
2. Case 2: $\alpha_1 = \overline{\alpha_2}$, $|\alpha_1| = 1$, $\alpha_3 \in (0, 1)$.
   Define $G_2 = \{(\alpha_1, \alpha_2, \alpha_3) \in (\overline{\Delta_1})^3 \mid \alpha_1 = \overline{\alpha_2},\ |\alpha_1| = 1,\ \alpha_3 \in (0, 1)\}$.

3. Case 3: $\alpha_1, \alpha_2 \in (-1,1)$, $\alpha_3 = 1$.
   Define $G_3 = \{\alpha_1, \alpha_2, \alpha_3 \in (\overline{\Delta_1})^3 \mid \alpha_1, \alpha_2 \in (-1,1),\ \alpha_3 = 1\}$.

We have (analogously to Theorem 4.5)

$$\mathcal{A}_0(\delta \mathbf{P}_3(\mathbf{R})) = \max_{(\alpha_1,\alpha_2,\alpha_3)\in G_1\cup G_2\cup G_3} \min_{|z|=1} \left| \frac{1}{(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)} \right|.$$

We first prove that if $(\alpha_1, \alpha_2, \alpha_3) \in G_2 \cup G_3$ then

$$\min_{|z|=1} \left| \frac{1}{(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)} \right| \leq \frac{1}{\sqrt{2}} < \frac{3+2\sqrt{2}}{8}.$$

1. $(\alpha_1, \alpha_2, \alpha_3) \in G_2$.
   Write $\alpha_1 = e^{i\theta}, \alpha_3 = r, s = \cos\theta$. Consider the points

   - $z_1 = -1$
   - $z_2 = 1$
   - $z_3 = i$
   - $z_4 = e^{i(\pi-\theta)}$.

   At these points $|(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)|^2$ attains the values:

   - $4(1+r)^2(1+s)^2$
   - $4(1-r)^2(1-s)^2$
   - $4s^2(1+r^2)$
   - $4s^2(4-3s^2)(1-s^2)$.

   From these estimates one easily deduces that

$$\max_{|z|=1} |(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)|^2 \geq 2.$$

2. If $(\alpha_1, \alpha_2, \alpha_3) \in G_3$ then

$$\max_{|z|=1} |(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)| \geq |1-i| = \sqrt{2}.$$

We now further investigate the case $(\alpha_1, \alpha_2, \alpha_3) \in G_1$.
 Write $\alpha_1 = re^{i\theta}, \alpha_3 = 1, s = \cos(\theta)$. Let $z = e^{i\psi}$. Then

$$|(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)|^2 = f_{rs}(x)$$

with

$$\begin{aligned} f_{rs}(x) &= (1-b_1)^2 + (b_0-b_2)^2 + 2x(b_2 + b_1 b_2 + b_0 b_1 - 3b_0) \\ &\quad + 2x^2(2b_0 b_2 + 2b_1) + 2x^3(4b_0) \end{aligned}$$

and

$$\begin{aligned} x &= \cos(\psi) \\ b_0 &= -r^2 \\ b_1 &= r^2 + 2rs \\ b_2 &= -1 - 2rs. \end{aligned}$$

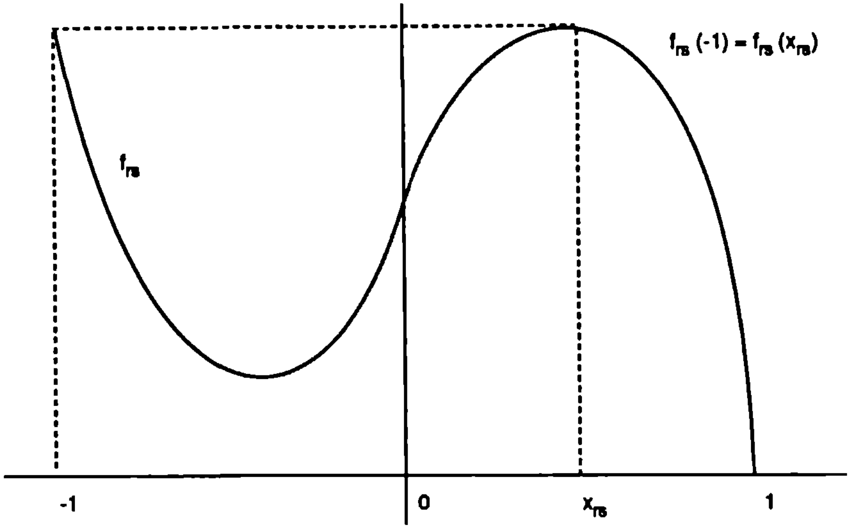

Figure 4.2

We first have to maximize $f_{rs}(x)$ over $x \in (-1, 1)$:
Let x_{rs} be the value of x for which $f_{rs}(x)$ attains its maximum.
As suggested by Figure 4.2, we have, in the maximizing case,

$$f_{rs}(x_{rs}) = f_{rs}(-1).$$

Let us assume the opposite, i.e. $f_{rs}(-1) < f_{rs}(x_{rs})$.
We find that the maximizing $p \in \delta\mathbf{P}_3^*(\mathbf{R})$ has the following properties:

- $p(1) = 0$,
- $p(z_0) = p(\overline{z_0}) = 0$ for some $z_0 \in \mathbf{C}$, $|z_0| > 1$, $z_0 \notin \mathbf{R}$,
- $|p(z_1)| = |p(\overline{z_1})| = 1$ for some $z_1 \in \Gamma_1$,
- $|p(z)| < 1 \quad (z \in \overline{\Delta_1} \setminus \{z_1, \overline{z_1}\})$.

Now let $q(z) = b_0 z + b_1 z^2 + b_2 z^3$ be such that $b_0, b_1, b_2 \in \mathbf{R}$ and

$$q(1) = p(1) = 0, \; q(z_1) = p(z_1)$$

(Such a q exists !). For $0 < \epsilon < 1$ define $g_\epsilon(z) = p(z) - \epsilon q(z)$. We claim that

1 For small ϵ: $\|g_\epsilon(z)\|_{\overline{\Delta_1}} < 1$.

2. For small ϵ: g_ϵ has all its zeros outside Δ_1.

From this a contradiction then follows, since the function

$$z \to \frac{g_\epsilon(z)}{\|g_\epsilon(z)\|_{\overline{\Delta_1}}}$$

is an element of $\delta\mathbf{P}_3^*(\mathbf{R})$ with larger constant term

Proof of claim 1:
(See Tsuji [30] and Figure 4.3).

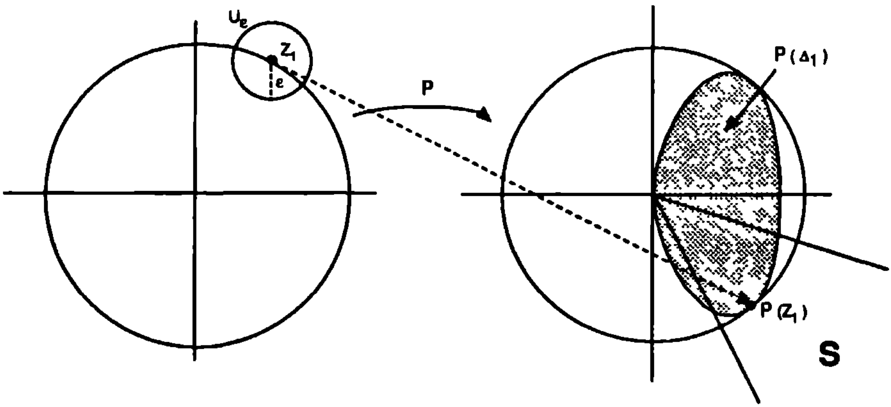

Figure 4.3

Let

$$\begin{aligned} U_\epsilon &= \{z \in \Delta_1 \mid |z - z_1| < \epsilon\}, \\ S &= \{w \in \mathbf{C} \mid \arg(w - p(z_1)) < \frac{\pi}{6}\}. \end{aligned}$$

For small δ_1 we have

$$p(U_{\delta_1}) \subseteq S, \quad q(U_{\delta_1}) \subseteq S$$

and since we may assume $1 \notin \overline{U_{\delta_1}}$ we find

$$|\delta_1 q(z)| < |p(z)| \quad (z \in U_{\delta_1}, \delta_1 \text{ small}).$$

For $w_1, w_2 \in S$ we have

$$|w_1 - w_2| < \max(|w_1|, |w_2|).$$

Therefore

$$|p(z) - \delta_1 q(z)| < |p(z)| \quad (z \in U_{\delta_1}).$$

Define $V_{\delta_1} = \{z \in \Delta_1 \mid \overline{z} \in U_{\delta_1}\}$.
A symmetry argument shows that

$$|p(z) - \delta_1 q(z)| < |p(z)| \quad (z \in V_{\delta_1}).$$

If $z \in \overline{\Delta_1} \setminus (U_{\delta_1} \cup V_{\delta_1})$, then $|p(z)| \leq c < 1$. Therefore if δ_2 is small enough

$$|p(z) - \delta_2 q(z)| < 1 \quad (z \in \overline{\Delta_1} \setminus (U_{\delta_1} \cup V_{\delta_1})).$$

The first claim now easily follows.

Proof of claim 2:
Let $r = \min(|\Im m\ z_0|, |z_0| - 1) > 0$. From Rouché's theorem it follows that for small ϵ, $p - \epsilon q$ will have a zero in both $\Delta_r(z_0)$ and $\Delta_r(\overline{z_0})$. Furthermore $p(1) - \epsilon q(1) = 0$. Thus we have three, i.e. *all* zeros outside $\overline{\Delta_1}$.

■

We now further assume $f_{rs}(x_{rs}) = f_{rs}(-1)$.
Write

$$\begin{aligned} \alpha &= 4r^2 \\ \beta &= 2rs(1 + r^2) \\ \gamma &= (r^2 + 2rs + 1)^2. \end{aligned}$$

From the facts

$$\begin{aligned} f_{rs}(-1) &= f_{rs}(x_{rs}) \\ f'_{rs}(x_{rs}) &= 0 \end{aligned}$$

we find

$$x_{rs} = \frac{\gamma + 2\alpha + 3\beta}{2\alpha + \beta}$$

and from this

$$-\alpha\gamma^2 + \gamma(-4\alpha^2 - 4\alpha\beta + \beta^2) + 4\beta^2(\alpha + \beta) = 0,$$

which implies $\gamma = -4(\alpha + \beta)$ or $\alpha\gamma = \beta^2$.

The case $\gamma = -4(\alpha + \beta)$ implies $x_{rs} = -1$, which implies that the maximum is attained only once. An argument similar to that which implied $f_{rs}(-1) = f_{rs}(x_{rs})$ then shows that this can never be a maximizing case. Therefore

$$\alpha\gamma = \beta^2.$$

From this we find the necessary condition

$$s = -\frac{1+r^2}{(1+r)^2}.$$

We then see that the corresponding value of $f_{rs}(x_{rs})$ is

$$2\{(1+b_1)^2+(b_0+b_2)^2\} = 4(1+2rs+r^2)^2 = \left(2\frac{(1+r^2)^2}{(1+r)^2}\right)^2.$$

Therefore the best choice for r is $r = \sqrt{2}-1$.
This yields the value $f_{rs}(x_{rs}) = (4-2\sqrt{2})^4$.

It now follows that

$$\mathcal{A}_0(\delta\mathbf{P}_3^*(\mathbf{R})) = \frac{1}{(4-2\sqrt{2})^2} = \frac{3+2\sqrt{2}}{8}.$$

■

Theorem 4.8

$$\mathcal{A}_1(\delta\mathbf{P}_3^*(\mathbf{R})) = \frac{10+7\sqrt{7}}{81}\sqrt{3}.$$

We have equality for the function

$$p: z \to \sqrt{3}\frac{4\sqrt{7}-29}{243}(z-1)^2\left(z+\frac{3+\sqrt{7}}{2}\right).$$

Proof:

Define G_1, G_2 and G_3 as in the previous theorem. Following the same reasoning as in that theorem we conclude that

$$\mathcal{A}_1(\delta\mathbf{P}_3^*(\mathbf{R})) = \max_{(\alpha_1,\alpha_2,\alpha_3)\in G_1\cup G_2\cup G_3} \min_{|z|=1} \frac{|\alpha_1+\alpha_2+\alpha_3|}{|(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)|}. \tag{4.1}$$

1. $(\alpha_1,\alpha_2,\alpha_3) \in G_1$. Let $\alpha_1 = re^{i\theta}, \alpha_3 = 1, \quad r\in(0,1), \theta\in[0,2\pi]$.

   Then (4.1) becomes

$$p_1(r,\theta) = \frac{|1+2r\cos(\theta)|}{\max_{\psi\in[0,2\pi]} |e^{i\psi}-1||e^{i\psi}-re^{i\theta}||e^{i\psi} \quad re^{-i\theta}|}.$$

   (a) Choose $e^{i\psi} = -1$. Then

$$p_1(r,\theta) \le \frac{|1+2r\cos(\theta)|}{2(r^2+2r\cos(\theta)+1)}.$$

(b) Choose $e^{\imath\psi} = i$. Then

$$p_1(r,\theta) \le \frac{|1+2r\cos(\theta)|}{\sqrt{2}\sqrt{(1-r^2)^2+4r^2\cos^2(\theta)}}.$$

If $1+2r\cos(\theta) \ge 0$, then the first choice shows $p_1(r,\theta) \le \frac{1}{2}$. Otherwise, the second choice shows that

$$p_1(r,\theta) \le \frac{1}{\sqrt{2}\sqrt{(1-r^2)^2+4}} < \frac{1}{2}.$$

2. $(\alpha_1, \alpha_2, \alpha_3) \in G_2$.
Let $\alpha_1 = e^{\imath\theta}, \alpha_3 = r, \quad r \in (0,1), \theta \in [0, 2\pi]$.
Then (4.1) becomes

$$p_2(r,\theta) = \frac{|r+2\cos(\theta)|}{\max\limits_{\psi\in[0,2\pi]} 2\sqrt{r^2-2r\cos(\psi)+1}\,|\cos(\psi)-\cos(\theta)|}. \tag{4.2}$$

(a) Suppose $r + 2\cos(\theta) \ge 0$.
Then
$$p_2(r,\theta) \le \frac{r+2\cos(\theta)}{2(r+1)(1+\cos(\theta))} \le \frac{r+2}{4(r+1)} \le \frac{1}{2}.$$

(b) Suppose $r + 2\cos(\theta) \le 0$.
Write $x = \cos(\psi)$. The denominator of (4.2),

$$2\sqrt{r^2-2rx+1}\,|x-\cos(\theta)|,$$

has two extremes, i.e.

$$x_1(r,\theta) = \cos(\theta),\ x_2(r,\theta) = \frac{r^2+r\cos(\theta)+1}{3r}.$$

i. Let $r > 2-\sqrt{3}$.
Then for all θ: $\frac{r^2+r\cos(\theta)+1}{3r} \le 1$.
Therefore

$$\begin{aligned} p_2(r,\theta) &\le \min_{x=-1,x=x_2(r,\theta)} \frac{-r-2\cos(\theta)}{2\sqrt{r^2-2rx+1}\,|x-\cos(\theta)|} \\ &= \min\left(\frac{-r-2\cos(\theta)}{2(1+r)(1+\cos(\theta))}, \frac{-3r\sqrt{3}(r+2\cos(\theta))}{2(r^2-2r\cos(\theta)+1)^{\frac{3}{2}}}\right). \end{aligned}$$

Note that both expressions are decreasing as function of $\cos(\theta)$. Therefore

$$p_2(r,\theta) \le 3r\sqrt{3}\frac{(2-r)}{2(1+r)^3} \quad (r \ge 2-\sqrt{3},\ r+2\cos(\theta) \le 0).$$

ii. Let $r \leq 2 - \sqrt{3}$.
Consider $\cos(\psi) = 1$. Then

$$p_2(r, \theta) \leq \frac{-r - 2\cos(\theta)}{2(1-r)(1-\cos(\theta))} \leq \frac{2-r}{4(1-r)} \leq \frac{3+\sqrt{3}}{8}.$$

3. $(\alpha_1, \alpha_2, \alpha_3) \in G_3$.
Let $\alpha_1 = r, \alpha_2 = s, \alpha_2 = 1, \quad -1 \leq r \leq s \leq 1$.
Then (4.1) becomes

$$p_3(r, s) = \frac{|1 + r + s|}{\max\limits_{\psi \in [0, 2\pi]} |e^{i\psi} - 1||e^{i\psi} - r||e^{i\psi} - s|}.$$

(a) Suppose $1 + r + s \leq -\frac{1}{2}$. Choose $e^{i\psi} = i$.
Then

$$\begin{aligned} p_3(r, s) &\leq \sqrt{\frac{(1+r+s)^2}{2(1+r^2)(1+s^2)}} \\ &\leq \sqrt{\frac{1}{2(\frac{25}{16})}} < \frac{1}{2}. \end{aligned}$$

(b) Suppose $-\frac{1}{2} \leq 1 + r + s \leq 0$.
Since

$$\max_{\psi \in [0, 2\pi]} |e^{i\psi} - 1||e^{i\psi} - r||e^{i\psi} - s| \geq 1$$

it follows that

$$p_3(r, s) \leq \frac{1}{2}.$$

(c) Suppose $1 + r + s \geq 0$. Choose $e^{i\psi} = -1$. Then

$$p_3(r, s) \leq \frac{1 + r + s}{2(1+r)(1+s)}.$$

i. If $rs > 0$, then

$$\frac{1+r+s}{2(1+r)(1+s)} \leq \frac{1}{2}.$$

ii. If $r > \sqrt{3} - 2$, then

$$\frac{1+r+s}{2(1+r)(1+s)} \leq \frac{2+r}{4(1+r)} \leq \frac{3+\sqrt{3}}{8}.$$

iii. $r \leq \sqrt{3} - 2, s \geq 0$. We shall show that for fixed r, the choice $s = 1$ is optimal.

Fix r. Rewrite $p_3(r,s)$ to

$$\frac{1+r+s}{\max\limits_{x\in[-1,1]}\sqrt{f_s(x)}}$$

where

$$\begin{aligned}
f_s(x) &= C_3(s)x^3 + C_2(s)x^2 + C_1(s)x + C_0(s)\\
C_3(s) &= -8rs\\
C_2(s) &= 4r(1+s^2) + 4s(1+r^2) + 8rs\\
C_1(s) &= -2(1+r^2)(1+s^2) - 4r(1+s^2) - 4s(1+r^2)\\
C_0(s) &= 2(1+r^2)(1+s^2).
\end{aligned}$$

The maximum of $f_s(x)$ can only be attained in two points, i.e. $x=-1$ or

$$x = x(s) = \frac{-C_2(s) - \sqrt{C_2^2(s) - 3C_1(s)C_3(s)}}{6C_3(s)}.$$

Here $x(s)$ is the smallest root of

$$g_s(x) = 3C_3(s)x^2 + 2C_2(s)x + C_1(s).$$

Some calculations shows that

- $g_s(-1) = -2((s^2+6s+1)(r^2+6r+1) - 16rs) \geq -2(r^2+6r+1) > 0,$
- $g_s(0) = -2((1+s^2)(1+r^2) + 2s(1+r^2)) \leq 0.$

Therefore, $x(s) \in [-1,0]$.
Since we want to show that $\frac{1+r+s}{\sqrt{f_s(x(s))}}$ is maximal in case $s=1$, it suffices to show that

$$\begin{aligned}
&\frac{\partial}{\partial s}\left(\frac{1+r+s}{\sqrt{f_s(x(s))}}\right) \geq 0\\
\Leftrightarrow\quad & 2f_s(x(s)) - (1+r+s)\frac{\partial}{\partial s}\left(f_s(x(s))\right) \geq 0\\
\Leftrightarrow\quad & 2[C_3(s)x^3(s) + C_2(s)x^2(s) + C_1(s)x + C_0(s)]\\
& -(1+r+s)[C_3'(s)x^3(s) + C_2'(s)x^2(s) + C_1'(s)x + C_0'(s)]\\
& -(1+r+s)x'(s)g_s(x(s)) \geq 0.
\end{aligned}$$

Note that $g_s(x(s)) = 0$. Rewrite the R.H.S. of the previous expression to

$$(s-1-r)\left|-8rx^3(s) + 4x^2(s)(1+r)^2 - 4(1+r^2)x(s)\right| \;+$$

$$+(2-2s(1+r))\left|4rx^2(s)-2(1+r)^2x(s)+2(1+r^2)\right|$$

i.e.

$$[-(s-1-r)2x(s)+2-2s(1+r)]\cdot\left[4r(x(s)-1)\left(x(s)-\frac{1+r^2}{2r}\right)\right].$$

Since we know that $-1 \leq x(s) \leq 0$ one easily deduces that this last expression is positive. Therefore, our claim is proved.
So, in case $r \leq \sqrt{3}-2, s \geq 0$, the best choice for s is $s=1$, in which case (4.1) becomes

$$-3r\sqrt{3}\frac{2+r}{2(1-r)^3}.$$

From the foregoing we now conclude that

$$\mathcal{A}_1(\delta\mathbf{P}_3^*(\mathbf{R})) \leq \max\left(\frac{3+\sqrt{3}}{8}, \max_{r\geq 2-\sqrt{3}} 3r\sqrt{3}\frac{2-r}{2(1+r)^3}\right).$$

It follows, by considering the optimal choice $r=3-\sqrt{7}$, that

$$\mathcal{A}_1(\delta\mathbf{P}_3^*(\mathbf{R})) = \frac{10+7\sqrt{7}}{81}\sqrt{3}.$$

■

From Theorems 4.7, 4.8 and Corollary 4.2 we now conclude that

$$\begin{aligned}
\mathcal{A}_0(\delta\mathbf{P}_3^*(\mathbf{R})) &= \frac{3+2\sqrt{2}}{8}\\
\mathcal{A}_1(\delta\mathbf{P}_3^*(\mathbf{R})) &= \frac{10+7\sqrt{7}}{81}\sqrt{3}\\
\mathcal{A}_2(\delta\mathbf{P}_3^*(\mathbf{R})) &= \frac{1}{2}\\
\mathcal{A}_3(\delta\mathbf{P}_3^*(\mathbf{R})) &= \frac{1}{2}.
\end{aligned}$$

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Samenvatting

In dit proefschrift worden verscheidene problemen betreffende de Taylor coëfficiënten van functies, analytisch in de eenheidsschijf, bekeken. Een van de belangrijke onderwerpen in de complexe functietheorie (van één variabele) is het bepalen van bovenbegrenzen voor de coëfficiënten van functies uit een gegeven klasse. Een beroemd voorbeeld hiervan is het vermoeden van Bieberbach, opgesteld in 1916, waarin wordt beweerd dat de Taylor coëfficiënten van iedere functie uit de klasse $\mathbf{S}$ voldoen aan de ongelijkheid $|a_n| \leq n$. In dit proefschrift wordt gekeken naar verscheidene klassen van functies die zowel begrensd zijn als een (vooraf gegeven) waarde ontwijken.

Indien bekend is dat in een klasse W alle elementen uniform begrensd zijn, dat betekent dat er een $M > 0$ is zodanig dat $|f(z)| \leq M$ ($z \in \Delta_1$, $f \in W$), dan geldt voor de coëfficiënten van iedere functie $f \in W$: $|a_n| \leq M$.

Indien van de functies in de klasse W verder niets geëist wordt, volgt meteen dat de zojuist gegeven bovengrens scherp is, dat betekent dat er een functie $f \in W$ (namelijk $f : z \to Mz^n$) bestaat waarvoor gelijkheid geldt. De vraag dringt zich dan al snel op wat er gebeurt indien we andere beperkingen aan W opleggen.

De klasse $\mathbf{B}^*$ is gedefinieerd door

$$\mathbf{B}^* = \{f \in H(\Delta_1) \mid 0 < |f| \leq 1\}.$$

De extra beperking hier is dus dat $f \in \mathbf{B}^*$ geen nulpunten heeft. Het probleem van het vinden van scherpe bovengrenzen voor de coëfficiënten lijkt niet eenvoudig. Krzyż uitte het vermoeden dat $|a_n| \leq \frac{2}{e}$ ($n \in \mathbf{N}$). In dit proefschrift wordt dit vermoeden bewezen voor de eerste vier coëfficiënten. Verder worden er nog diverse andere klassen van functies bekeken die te maken hebben met de klasse $\mathbf{B}^*$.

Curriculum Vitae

De auteur van dit proefschrift werd geboren op 30 juni 1960 te Wijchen. In 1978 behaalde hij het VWO diploma aan het Titus Brandsma Lyceum te Oss. Hij studeerde vervolgens wiskunde aan de Katholieke Universiteit Nijmegen en behaalde daar in maart 1982 zijn kandidaatsexamen, gevolgd door het doctoraalexamen (cum laude) in juni 1984. Vanaf augustus dat jaar tot september 1988 verrichtte hij promotieonderzoek onder leiding van Prof. A.C.M. van Rooij en in samenwerking met Dr. R.A. Kortram op het gebied van complexe functies, wat leidde tot dit proefschrift. Sinds januari 1989 is hij werkzaam als docent Wiskunde en Informatica aan de Hogeschool Eindhoven.

STELLINGEN

behorende bij het proefschrift
Coefficient estimates for bounded nonvanishing functions
van R.J.P.M. Ermers

1. Zij H een Hilbertruimte, B_H de eenheidsbol van H, en laat $K \subset \mathcal{L}(H)$ gedefinieerd zijn door:

   $T \in K \Leftrightarrow$ Voor iedere partiële isometrie $J \in \mathcal{L}(H)$ met beginruimte $\overline{R_T}$ waarvoor $J(T(B_H)) \subset T(B_H)$ geldt: $J(T(B_H)) = T(B_H)$.

   Dan geldt:
   $$T \in K \Leftrightarrow T^* \in K.$$

   (R. Ermers, *Het beeld van een compacte operator* (doctoraalscriptie 1984))

2. Zij $n \in N$, $r_1 \geq r_2 \geq \cdots \geq r_n$, $s_1 \geq s_2 \geq \cdots \geq s_n$, $r_i, s_i \in \mathbf{N}$; de r_i zijn de rijsommen en de s_i de kolomsommen van een $(0,1)$ matrix. Zij $T = (t_{i,j})$ de bijbehorende structuurmatrix waar
   $$t_{i,j} = i \cdot j + \sum_{k>i} r_k - \sum_{l \leq j} s_l, \qquad i,j = 0,1,\ldots,n$$
   en T^* de matrix waarvoor geldt: $E_{n+1} T^* E_{n+1}^T = T$, met E_{n+1} de $(n+1) \times (n+1)$-matrix
   $$\begin{pmatrix} 1 & & & \\ 1 & 1 & & \emptyset \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$
   Veronderstel dat Rang $T^* = 3$. Dan zijn de drie van nul verschillende eigenwaarden van T^* reëel.
   (R. Ermers and B. Polman, *On the eigenvalues of the structure matrix of matrices of zeros and ones*, Linear Algebra and its Applications **95**(1987), 17-41)

3. Met dezelfde notatie als in stelling 2, veronderstel dat Rang $T = 2$. Laat λ_1 en λ_2 de twee eigenwaarden van T ongelijk nul zijn. Dan geldt
   $$n \leq \lambda_1, \lambda_2 \leq \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{12}n + \frac{1}{2}\sqrt{\frac{1}{36}(n+1)^2\, n\,(4n^3 - 8n^2 - 11n + 24)}.$$
   en gelijkheid is mogelijk.
   (R. Ermers and B. Polman, *On the eigenvalues of the structure matrix of matrices of zeros and ones*, Linear Algebra and its Applications **95**(1987), 17-41)

4. Voor iedere $n \in \mathbf{N}$ bestaat er een functie $f_n \in \mathbf{B}$, $f_n(z) = \sum_{j=0}^{\infty} a_j z^j$ met de volgende eigenschappen:

   - $|a_n| > 1 - \frac{1}{2n}$,
   - $\sup_{z \in \Delta_1} \left| \sum_{j=0}^{n-1} a_j z^j \right| \geq \frac{1}{2}$.

5. Zij $\mathbf{M}$ de klasse van functies f, meromorf en univalent in Δ_1, genormeerd door $f(0) = 0, f'(0) = 1$;
   Zij $\mathbf{S}$ de subklasse van $\mathbf{M}$ bestaande uit die functies, die de waarde ∞ ontwijken.
   Voor $f \in \mathbf{M}$ zij $[f] = \left\{ g \in \mathbf{M} \mid g = \frac{wf}{w-f}, w \in C^* \setminus \{0\} \right\}$. Dan geldt:
   Voor iedere $f \in \mathbf{M}$ is er een $h \in [f] \cap \mathbf{S}$ zodanig dat

$$h(z) = \sum_{n=1}^{\infty} a_n z^n, |a_2| \leq 1, |a_3| \leq 1.$$

6. Met dezelfde notatie als in stelling 5:
   Voor iedere $f \in \mathbf{M}$ en iedere $k \in \mathbf{N}$, is er een $h \in [f] \cap \mathbf{S}$ zodanig dat

$$h(z) = \sum_{n=1}^{\infty} a_n z^n, |a_k| \leq 1.$$

7. Het programmeren van een omvangrijke én bugvrije computerapplicatie is even moeilijk als het schrijven van een foutloos proefschrift (Cf.1: Super48. Uitgever: Eiland Software; Cf.2: StFinder. Uitgever: Commedia Amsterdam).

8. Het is wijzer er op te wijzen dat de wijze waarop men met behulp van Pascal wijzers onderwijst, je niet veel wijzer maakt.

9. Een verwijzing naar het proefschrift getiteld *Differences and Similarities in the Use of the First Person Singular by English and American Humorous Writers between 1850 and 1950* kwam eerder dan de auteur verwachtte.